THE APRIL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and fifty-eighth regular meeting of the Society was held in New York City on Saturday, April 27, 1912. The attendance at the two sessions included the following fifty-two members:

Mr. J. W. Alexander, Professor R. C. Archibald, Mr. H. Bateman, Mr. A. A. Bennett, Professor W. J. Berry, Professor G. D. Birkhoff, Professor E. W. Brown, Professor B. H. Camp, Dr. Emily Coddington, Professor F. N. Cole, Dr. G. M. Conwell, Professor J. L. Coolidge, Professor E. S. Crawley, Dr. H. B. Curtis, Dr. L. S. Dederick, Dr. L. L. Dines, Mr. E. P. R. Duval, Professor H. B. Fine, Professor T. S. Fiske, Professor W. B. Fite, Professor O. E. Glenn, Mr. G. H. Graves, Professor C. C. Grove, Professor L. A. Howland, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Dr. J. K. Lamond, Dr. N. J. Lennes, Professor W. R. Longley, Dr. H. F. MacNeish, Dr. Emilie N. Martin, Dr. E. J. Miles, Dr. H. H. Mitchell, Professor C. L. E. Moore, Dr. R. L. Moore, Professor W. F. Osgood, Mr. E. S. Palmer, Dr. Anna J. Pell, Professor A. D. Pitcher, Dr. H. W. Reddick, Professor R. G. D. Richardson, Dr. J. E. Rowe, Professor L. P. Siceloff, Mr. C. G. Simpson, Professor D. E. Smith, Professor Sarah E. Smith, Dr. W. M. Smith, Professor Henry Taber, Professor J. H. Tanner, Professor Oswald Veblen, Miss M. E. Wells.

The President of the Society, Professor H. B. Fine, occupied the chair. The Council announced the election of the following persons to membership in the Society: Miss S. R. Benedict, Smith College; Mr. C. E. Fisher, Rhode Island Normal School; Dr. T. H. Gronwall, Chicago, Ill.; Mr. L. A. Hopkins, University of Michigan; Dr. A. J. Kempner, University of Illinois; Mr. V. C. Poor, University of Michigan; Mr. R. B. Stone, Harvard University; Mr. K. P. Williams, Princeton University. Seven applications for membership in the Society were received.

It was decided to hold the annual meeting of the Society this year at Cleveland, Ohio, in affiliation with the American Association for the Advancement of Science. The winter meeting of the Chicago Section will be merged in the annual meeting. Owing to President Fine's absence abroad, the delivery of his presidential address will be postponed to the annual meeting of 1913.

The following papers were read at the April meeting:

(1) Dr. R. L. MOORE: "Concerning Jordan curves in non-metrical analysis situs."

(2) Dr. J. K. LAMOND: "Improper multiple integrals over iterable fields."

(3) Professor L. A. HOWLAND: "Binary conditions for singular points on a cubic."

(4) Professor B. H. CAMP: "Certain integrals containing parameters."

(5) Dr. S. LEFSCHETZ: "On the V_3^3 with five nodes of the second species in S_4.

(6) Dr. E. R. MARSHALL: "A labor-saving device in computation."

(7) Professor G. D. BIRKHOFF: "The reducibility of maps."

(8) Professor G. D. BIRKHOFF: "A determinant formula for the number of ways of coloring any map."

(9) Professor OSWALD VEBLEN: "An application of modular equations in analysis situs."

(10) Dr. H. B. PHILLIPS and Professor C. L. E. MOORE: "A geometric use of matrices."

(11) Dr. H. B. PHILLIPS and Professor C. L. E. MOORE: "A theory of linear distance and angle."

(12) Professor L. P. SICELOFF: "Sylow subgroups in groups whose orders are of certain special forms."

(13) Professor A. D. PITCHER: "Concerning the continuity and convergence of functions of a general variable."

(14) Professor W. R. LONGLEY: "Proof of a theorem due to Picard."

(15) Mr. A. R. SCHWEITZER: "Remark on a functional equation."

(16) Mr. A. R. SCHWEITZER: "Theorems on functional equations."

(17) Dr. DUNHAM JACKSON: "On approximation by trigonometric sums and polynomials (second paper)."

(18) Dr. N. J. LENNES: "Concerning Van Vleck's non-measurable set."

(19) Dr. N. J. LENNES: "Concerning infinite polygons and polyhedrons."

The papers of Dr. Lefschetz, Professor Pitcher, Mr. Schweitzer, Dr. Jackson, and Dr. Lennes were read by title. Dr. Lefschetz's paper appeared in the May BULLETIN. That of Professor Longley is published in the present number of the BULLETIN. Abstracts of the other papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. In the *Transactions* of this Society, volume 6, pages 83–98, Professor Veblen has given a definition of a simple plane curve lying in a plane which satisfies Axioms I–VIII, XI of his system of axioms for geometry. Let V denote such a plane. Dr. Moore wishes to show that:

I. There is a one to one correspondence between the points of V and a part H of an ordinary number plane, this correspondence being (in an easily defined sense) continuous. Thus each point of V is represented by a pair of coordinates (x, y).

II. A simple curve as defined by Veblen may be represented by a pair of equations $x = f_1(t)$, $y = f_2(t)$, where $f_1(t)$ and $f_2(t)$ are continuous functions of the numerical parameter t.

III. The plane V is an Ebene as defined by Hilbert in his article "Ueber die Grundlagen der Geometrie" (*Mathematische Annalen*, volume 56, page 383).

2. Dr. Lytle has shown that the fundamental relation

$$\underline{\int}_{\mathfrak{A}} f \leqq \underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} f \leqq \overline{\int}_{\mathfrak{B}} \overline{\int}_{\mathfrak{C}} f \leqq \overline{\int}_{\mathfrak{A}} f,$$

where the integrals are proper integrals, holds for a class of fields which he calls iterable. Using a new definition of an improper multiple integral, which is due to Professor Pierpont, and considering unlimited functions defined, in general, over iterable fields, Dr. Lamond gives conditions under which the above fundamental inequalities hold. In some cases the function f may have points of infinite discontinuity at an everywhere dense set.

3. The cubic plane curve has been exhaustively studied by use of the theory of ternary forms. Conditions necessary and sufficient for the existence of one or more double points, of a triple point, etc., have been derived in terms of ternary in-

variants, chiefly by Aronhold, Gordan, Clebsch, and Gundelfinger.

The theory of binary forms, however, may also be applied to the study of the cubic. Compare, for example, the treatment of the inflexions of a general cubic by Clebsch in his Theorie der binären Formen. In Professor Howland's paper conditions for the existence of a double point (node or cusp) and that of a triple point are derived, expressed in terms of the simultaneous invariants of two binary forms.

4. In a recent paper (*Annales de la Faculté de Toulouse*, series 3, volume 1 (1909), pages 25–128) Lebesgue has studied the integrals

$$\int_a^b f(t)\phi(t, n)dt, \quad \int_a^b f(t)\phi(t - x, n)dt.$$

His fundamental theorems have been generalized by Professor Camp so as to apply to functions of m variables and k parameters, defined over limited fields of very general natures. Moreover, corresponding theorems may be obtained for certain classes of functions not considered by Lebesgue. Applications of the results are found in the theory of the development of an arbitrary function in series of normal functions.

6. In insurance computations multiplications or divisions of a series of numbers by a constant are of frequent occurrence, in which the results are taken to the nearest unit, or 1st, 2d, or 3d, etc., decimal. A new method for obtaining the results of such serial multiplication or division with a minimum of labor is set forth in Dr. Marshall's paper. The method is based upon the following consideration:

Let $x_{0,1}, x_{0,2}, \cdots, x_{0,k}, \cdots; x_{1,1}, x_{1,2} \cdots, x_{1,k}, \cdots; \cdots, x_{i,1}, x_{i,2}, \cdots, x_{i,k}, \cdots; \cdots$ represent the required quotients, where the first subscripts indicate their nearest integers, and let $a_{0,1}, a_{0,2}, \cdots$ represent the corresponding variable dividend, while m is the constant divisor, so that $x_{i,k} = a_{i,k} \div m$. Then we have $x_{0,k} < .5 \leqq x_{1,k} < 1.5 \leqq x_{2,k} < 2.5 \cdots$.

Hence, in general, $a_{i,k} < (i + .5)m \leqq a_{i+1,k} < (i + 1.5)m$ (where the $=$ of the double symbol may hold only for $k = 1$). Hence all the values of $a_{i+1,k}$ giving the integer $i + 1$ lie between $(i + .5)m$ and $(i + 1.5)m$. For multiplication the reciprocal of m, obtained to a certain degree of approximation, is used instead of the constant multiple m itself.

The author illustrates the process by a concrete example from insurance practice, and proves that on the arithmometer, which is an especially adapted calculating machine for this kind of computation, the saving of labor resulting from the use of this method is between 75 and 90 per cent. In the second part of his paper the author shows how to extend his method to the finding of quotients and products in which an approximation of a number of decimals is required and proves by actual illustration that his method can be advantageously used in a large field of actuarial practice, comprising divisions with moderately large differences of results, not exceeding two or three figures. It should be stated that the method is of most advantage only in cases where the differences of the results are smaller than those of the given series of multipliers or dividends. The illustrations of the second part of the paper are taken from Mr. George King's conversion and valuation tables appended to his paper, "On policies with deferred participation in profits, etc.," read before the Faculty of Actuaries in Scotland and printed in the *Transactions* of the Faculty, volume 5, part 9, number 83 (1911).

In conclusion the author shows how his method could also be used with considerable advantage in the computation of deferred and temporary annuities.

7. It is well-known that the problem of coloring a given map in four colors is reducible to that of coloring other maps of fewer regions in case the given map has (1) any multiply connected regions; (2) more than three regions meeting at any point; (3) any region bounded by less than five others; these facts comprise nearly all that has been done toward the reduction of maps.

By defining reducibility in a technical sense in which a principle due to Kempe is involved, Professor Birkhoff shows that numerous other configurations occur only in reducible maps; thus any ring of less than five regions and a ring of five regions not bounding a single region are proved to occur only in reducible maps.

8. This note by Professor Birkhoff on a formula for the

number of ways of coloring a map in any number of colors will appear in the *Annals of Mathematics*.

9. In the note by Professor Veblen it is shown that the circuits and systems of circuits in a complex of lines (in the sense of analysis situs) may be represented by the solutions of a system of linear equations, modulo two. The operations of linearly combining these equations or their solutions are interpreted geometrically. A similar theory holds for complexes of n dimensions. The equations may be applied in various problems of analysis situs, including the four color problem and the proof of Euler's theorem.

10. A point A in n dimensions is determined by $n+1$ homogeneous coordinates and may be represented by the matrix $||a_1 \cdots a_{n+1}||$. A linear function $\lambda A + \mu B$ of two points is represented by the one-rowed matrix whose elements are $\lambda a_i + \mu b_i$. A line AB is represented by a two-rowed matrix $[AB]$ formed by writing the coordinates of A above the coordinates of B. The elements of this matrix are the two-rowed determinants obtained from it and not the individual coordinates. The sum of two such matrices is formed by adding the corresponding elements. This new matrix is not, in general, expressible as a two-rowed matrix of $n+1$ columns. Similarly, planes are represented by three-rowed matrices $[ABC]$, three-spaces by four-rowed, etc.

These quantities $[AB]$, $[ABC]$, etc., may be considered as products obeying the laws of Grassmann's outer product, and are called progressive. The operation consists of writing the coordinates of the first factor in the first row, the coordinates of the second factor in the second row, etc., to form a new matrix.

Hyperplanes are used as coordinates in the same way and the products thus obtained are called regressive. If the number of points representing the factors of a product is greater than $n+1$, the factors are expressed in plane coordinates and multiplied regressively. One of the chief aims of Dr. Phillips and Professor Moore is to express this regressive product in terms of points, thus obtaining Grassmann's reduction formula.

11. In this paper Dr. Phillips and Professor Moore define

the distance between two points, in three dimensions, by the following properties: (1) If one point is fixed and the distance held constant, the other describes a plane. (2) Distances between points on a line have the additive property. It is found that the distance is expressible in terms of a fundamental system consisting of a plane and a linear complex. Distance will then be invariant under a six-parameter group of collineations leaving the fundamental system fixed. The angle between two planes is defined as the dual of the distance between two points. In this case the fundamental system consists of a point and linear complex. In order that distances and angles be left invariant by the same six-parameter group of collineations, the complexes are taken as the same and the point as the pole of the plane with respect to the complex. In terms of this fundamental system angle between lines, distance from point to line, etc., are defined. Formulas for the area of a triangle and volume of a tetrahedron are also obtained. These results are generalized for n dimensions.

12. In this paper Professor Siceloff studies restrictions on the structure of a group corresponding to certain restrictions on the form of its order. He proves in the first theorem that a group of order $(kp + 1)p$ cannot have the full number, $kp + 1$, of subgroups of order p if p is greater than some prime-power factor of $kp + 1$. The second theorem states that in a group whose order has only one factor (other than unity) $\equiv 1 (\text{mod } p)$, the elements common to two Sylow subgroups of order p^a form an invariant subgroup. The third theorem deals with an important special case under the hypothesis of theorem 2.

13. In his Introduction to a Form of General Analysis and in his lectures at the University of Chicago Professor E. H. Moore has given a theory of developments of a general range and has indicated the beginning of a general theory of continuity and convergence of functions relative to a development of the range of the functions. Professor Pitcher secures the ordinary properties of continuous and convergent functions by means of simple postulates on the development of the range of the functions.

15. The purpose of Mr. Schweitzer's first paper is to explain

a reference to *Crelle's Journal* given on page 300 of his paper, "On a functional equation," BULLETIN, volume 18, pages 299–302. The relation $\phi(x,y) = \Omega\{\psi(x) - \psi(y)\}$ can be written $\Omega^{-1}\{\phi(x,y)\} = \psi(x) - \psi(y)$, and the preceding reference suggests $\Omega^{-1}\{x\phi(y) - y\phi(x)\} = \psi(x) - \psi(y)$ as a correlative of the equation of Abel, *Crelle,* volume 2, page 389. Corresponding to Abel, l. c., page 386, one gets a particular case of the relation $\chi\{\phi(x,y)\} = \chi(x) - \chi(y)$ by assuming that $\Omega^{-1}(x) = \psi(x)$ in the preceding correlative. It is of interest to note here that C. Lottner, *Crelle,* volume 46, pages 367, 368, etc., has discussed a special case of $\Omega^{-1}\{\phi(x,y)\} = \psi(x) + \psi(y)$ in relation to the preceding paper of Abel.

16. In connection with his recent article in the BULLETIN, March, 1912, page 300, Mr. Schweitzer proves the following theorems:

I. If $\phi[x, \phi(y, z)] = \phi[y, \phi(x, z)]$ and $\phi[\phi(y, z), z] = y$ and $\phi[\phi(y, z), y] = z$, then there exists a function χ_ϕ such that $\phi(x,y) = \chi_\phi^{-1}\{\chi_\phi(x) + \chi_\phi(y)\}$ and $\phi(x,y) = \chi_\phi^{-1}\{\chi_\phi(x) - \chi_\phi(y)\}$.

II. If $\phi[y, \phi(x, z)] = \phi[z, \phi(x, y)]$ and $\phi[\phi(y, z), z] = y$ and $\phi[\phi(y, z), y] = z$, then there exists a function χ_ϕ such that $\phi(x, y) = \chi_\phi^{-1}\{\chi_\phi(x) - \chi_\phi(y)\}$ and $\phi(x,y) = \chi_\phi^{-1}\{\chi_\phi(x) + \chi_\phi(y)\}$.

17. Following out further the line of investigation on which he reported at the February meeting, Dr. Jackson is led to the following conclusions: If $f(x)$ is a function of period 2π which satisfies a Lipschitz condition with coefficient λ, that is, if it is always true that

$$|f(x_2) - f(x_1)| \leqq \lambda | x_2 - x_1 |,$$

then $f(x)$ can be approximately represented for all values of x by a trigonometric sum of order n or lower with a maximum error not exceeding $3\lambda/n$, for all positive integral values of n. If $f'(x)$ satisfies a Lipschitz condition with coefficient λ, the maximum error of such an approximation for $f(x)$ can be made not to exceed $20\lambda/n^2$. If $f(x)$ satisfies the same conditions, except as to periodicity, in a closed interval of length 1, it may be represented in this interval by a polynomial of the nth or lower degree with an error not greater than $\frac{3}{2}\lambda/n$ or $10\lambda/n^2$ respectively. If $f(x)$ satisfies the first of the conditions above, it differs from the partial sum of its Fourier series, to terms of the nth order, by not more than $6\lambda \log n/n$, provided $n \geqq 5$.

18. In this paper Dr. Lennes describes in detail the construction of a set which is non-measurable in the sense of Lebesgue. The general method is the same as that used by Van Vleck (*Transactions,* volume 9, page 237), but each step is made to depend upon explicitly formulated postulates.

19. In two papers published in the *American Journal of Mathematics,* volume 33, Dr. Lennes formulated a certain body of theorems on polygons and polyhedrons. With one exception these were confined to figures having a finite number of sides or faces. In the present paper many of the results obtained in these papers are extended to polygons and polyhedrons having an infinite number of sides or faces.

F. N. Cole,
Secretary.

PROOF OF A THEOREM DUE TO PICARD.

BY PROFESSOR W. R. LONGLEY.

(Read before the American Mathematical Society, April 27, 1912.)

Consider the ordinary differential equation of the first order and second degree

$$Ap^2 + 2Bp + C = 0 \qquad (p = dy/dx), \tag{1}$$

in which the coefficients are power series in x and y vanishing when $x = y = 0$

$$A = ax + a_1y + \cdots, \quad B = bx + b_1y + \cdots, \quad C = cx + c_1y + \cdots. \tag{2}$$

Picard has proved that in the general case* every integral curve (real curve in the cartesian plane) of equation (1) which comes infinitely near the origin, actually reaches the origin with a determinate tangent. The proof† is based upon the existence of an analytic integral curve passing through the origin, a condition which is satisfied in the general case. But such a curve does not always exist, and the following proof, which

* By the general case it is meant that there exists no particular relation of equality among the coefficients of (2).

† Picard, *Comptes Rendus,* vol. 120 (1895) p. 524; *Math. Annalen,* vol. 46 (1895), p. 521; Traité d'Analyse, vol. 3, pp. 217–225.