THE FIFTH REGULAR MEETING OF THE SOUTHWESTERN SECTION.

THE fifth regular meeting of the Southwestern Section of the Society was held at Washington University, St. Louis, Missouri, on Saturday, December 2, 1911. About twenty-five persons were present, including the following members of the Society:

Professor W. C. Brenke, Dr. A. R. Crathorne, Professor Ellery W. Davis, Dr. Otto Dunkel, Professor H. C. Harvey, Professor E. R. Hedrick, Professor G. O. James, Professor O. D. Kellogg, Professor A. M. Kenyon, Mr. W. C. Krathwohl, Dr. S. Lefschetz, Professor H. L. Rietz, Professor W. H. Roever, Professor J. N. Van der Vries, Professor C. A. Waldo.

The morning session opened at 10 A.M. and the afternoon session at 3 P.M., Professor Roever presiding. Lawrence, Kansas, was decided upon as the next place of meeting and the following program committee was elected: Professors Van der Vries (chairman), Davis, and Kellogg (secretary). The members presentt atended a smoker at the University Club on the evening before the meeting and dined together between the sessions in the dining room of Tower Dormitory.

The following papers were presented at this meeting:

(1) Professor R. D. CARMICHAEL: "Generalizations of Euler's φ-function with applications to abelian groups."

(2) Dr. OTTO DUNKEL: "A necessary condition for the reality of the roots of an algebraic equation."

(3) Professor O. D. KELLOGG: "Note on periodic functions with derivatives of all orders."

(4) Dr. S. LEFSCHETZ: "Two theorems on conics."

(5) Professor M. B. WHITE: "The dependence of the focal point on curvature in space problems in the calculus of variations."

(6) Professor IVA ERNSBERGER: "The imaginary parabola" (preliminary communication).

(7) Professor HEDRICK and Dr. INGOLD: "Axioms of line geometry."

(8) Professor W. H. ROEVER: "The deviation of falling bodies for a distribution not of revolution."

(9) Professor E. R. HEDRICK: "The concept of limit."

(10) Professor G. O. JAMES: "Comparison of methods of adjusting outstanding differences in the motions of the four inner planets."

(11) Dr. A. B. FRIZELL: "On a subset of the terms in the infinite determinant having the potency of the continuum."

Professor Ernsberger was introduced by Professor Davis. In the absence of the authors Professor Carmichael's paper was read by title and the papers of Professor White and Dr. Frizell were presented by Professor Van der Vries. Abstracts of the papers follow.

1. Among the many generalizations which have been given of Euler's φ-function, particular mention may be made of those due to Schemmel, Lucas, Vahlen, Cohen, and Zsigmondy. The last (*Monatshefte für Mathematik und Physik*, volume 7 (1896), pages 185–289) especially, is one of far-reaching importance. It comes out as an application of a general theory of abelian groups, which Zsigmondy develops. The object of Professor Carmichael's paper is to define in a new way Zsigmondy's generalization of the φ-function, and to develop its fundamental properties from the point of view of the new definition. As immediate corollaries of this theory of the generalized φ-function, one obtains Zsigmondy's fundamental results in the theory of abelian groups by means which are essentially simpler than those employed by Zsigmondy.

2. A necessary condition that all the roots of the equation $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0$ be real is that the series of determinants of even order

$$\begin{vmatrix} a_0 & a_1 \\ a_1 & a_2 \end{vmatrix}, \qquad \begin{vmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & a_5 \\ a_3 & a_4 & a_5 & a_6 \end{vmatrix}, \quad \cdots$$

have signs alternately $-$ and $+$. This result which is obtained by Dr. Dunkel by the methods employed in a previous paper in the *Annals of Mathematics*, series 2, volume 10 (1908), page 46, bears some analogy to a sufficient condition already given by Professor E. B. Van Vleck in the same journal, series 2, volume 4 (1903), page 191.

3. If $f(x)$ together with its derivatives of all orders has the period 2π, then either $\max|f^{(k)}(x)| > Ae^{\lambda k}$ for all but a finite number of values of k, A and λ being positive constants, or else $f(x)$ is a terminating Fourier series. Professor Kellogg's paper gives a simple proof of this fact, and mentions some inferences.

4. In this paper Dr. Lefschetz treats of two correlated theorems on conics. (1) When two complete quadrilaterals, circumscribed to the same conic S have a common diagonal, their vertices exterior to it are on a conic Γ. Let D be the common diagonal, A, B, A', B' the vertices on it. It is shown both analytically and synthetically that for a given conic the vertices A, B are conjugate elements of an involution on D, and conversely that to each involution on D there corresponds one and only one conic Γ. These theorems can be deduced easily from others given by Poncelet, although the proofs of the present paper are completely different. It is further shown that the system S, Γ is projectively ∞^1, and the relation satisfied by the invariants of the two conics is also given. (2) When the tangents at three of the six common points of a conic and a triangle go through the opposite vertices, they all do, the triangle being then self-polar with respect to the conic. Here also both analytic and synthetic proofs are given, and a third proof based upon projective considerations. This theorem is equivalent to the following well known proposition in the geometry of rational plane quartics: if a rational plane quartic has three flecnodes, it has necessarily three biflecnodes. The paper will be offered for publication to the *Annals of Mathematics*.

5. If an arc C_{01} which joins a space curve L and a fixed point 1 minimizes the integral $J = \int f(x, y, y', z, z')dx$ with respect to other curves joining L with 1, there will in general be a focal point 2 on the curve of which C_{01} is a part at which this minimizing property ceases. Professor White shows that the position of the focal point is a function of the curvature of L at 0 and discusses its properties by means of the second variation. The paper will be offered to the *Transactions*.

7. In this paper Professor Hedrick and Dr. Ingold give a system of axioms for line geometry based upon the fundamental ideas of line and intersection. They are able with a com-

paratively simple set of assumptions, to prove the fundamental propositions of line geometry. The assumptions may also be made the basis of a treatment of projective geometry by defining the words point, plane, etc., in terms of the fundamental ideas mentioned above.

8. In order to be able to compare the results of experiment with those of theory, the following definitions are taken: Let P_0 denote the point from which the body falls, c the path of the falling body, and d the locus of plumb-bobs of all plumb-lines which are supported at P_0. The curve d pierces a level surface in a point D. Let π denote the tangent plane to this level surface at D, and C the point in which the curve c pierces the plane π. Let α and β denote the east-and-west and the north-and-south lines respectively of the plane π which pass through D. The distance of C south of α is the southerly deviation of the falling body and that of C east of β is the easterly deviation of the falling body. If h represents the height of P_0 above π and W the potential function of the field of force in which the plumb-line is in equilibrium, it is possible to express the deviations just defined in terms of h and the first and second derivatives of W at P_0. These expressions are deduced in Professor Roever's paper. The quantities h and g (the only first derivative of W which enters) can be measured. The second derivatives of W which enter can be found experimentally by a method due to Baron Eötvös. (Encyklopädie der mathematischen Wissenschaften, Band VI, 1, B, Heft 2, § 23, page 166.) Hence it is not necessary to assume a mathematical expression for W. Since experiments have been, and are likely to be, made in localities where the local deviation of the gravitational field is considerable (due to the proximity of mountains) the advantage of this method is apparent.*

9. In the paper presented by Professor Hedrick it is shown that a material simplification of the system of axioms proposed by Steinhaus in a recent number of the *Mathematische Annalen* for the concept "limit" leaves that concept defined uniquely on convergent sequences, and permits extension to divergent sequences in the Cesàro sense. It is shown that the concept

* The southerly deviation for a distribution of revolution has already been treated by Professor Roever in a paper in the *Transactions*, vol. 12, No. 3, pp. 335–353.

"limit" may be defined for general systems of objects in an analogous manner, without excluding the possibility of special examples of the Cesàro type.

10. The nodes and perihelia of the four inner planets, notably Venus and Mars, present certain unexplained motions in the Newtonian mechanics. The note of Professor James compares the secular changes in the elements of these planets produced by the uniform rotation of the empirical about the inertial system of reference with the corresponding changes brought about by the use of the Minkowskian law of attraction instead of the Newtonian.

11. In this paper Dr. Frizell shows that a one-to-one relation exists between the continuum and a set of terms in the expansion of an infinite determinant whose elements are restricted to the principal diagonal and two adjacent diagonals.

O. D. KELLOGG,
Secretary of the Section.

SERIES OF LAPLACE'S FUNCTIONS.

BY PROFESSOR B. H. CAMP.

(Read before the American Mathematical Society, October 28, 1911.)

THE most important theorem on the validity of the expansion of an arbitrary function in a series of Laplace's functions has been proved by Jordan in his Cours d'Analyse, second edition, volume 2, page 252. The conditions there stated are that the given function be continuous on the surface of the sphere within some small circle about the point at which the expansion is made, and that it have limited variation along every great circle through this point.

The object of the present paper is to correct an error in Jordan's theorem, and to furnish new conditions sufficient for the validity of these expansions. To the conditions announced by Jordan should be added the requirements that the values of the variations be all less than some fixed number, and that these variations be "uniform with respect to all great circles through the point." His error is discussed in a remark following Corollary 2.