equations (2) it follows that

$$\begin{array}{l}\frac{\partial f_1}{\partial y_1}\frac{\Delta y_1}{\Delta x_1}+\frac{\partial f_1}{\partial y_2}\frac{\Delta y_2}{\Delta x_1}+\cdots+\frac{\partial f_1}{\partial y_n}\frac{\Delta y_n}{\Delta x_1}+\frac{\partial f_1}{\partial x_1}=0,\\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\ \frac{\partial f_n}{\partial y_1}\frac{\Delta y_1}{\Delta x_1}+\frac{\partial f_n}{\partial y_2}\frac{\Delta y_2}{\Delta x_1}+\cdots+\frac{\partial f_n}{\partial y_n}\frac{\Delta y_n}{\Delta x_1}+\frac{\partial f_n}{\partial x_1}=0,\end{array}$$

where the arguments of the derivatives $\partial f_i/\partial x_1$ have the form $x + \theta_i'\Delta x; y + \Delta y$. Hence as Δx_1 approaches zero the quotients $\Delta y_i/\Delta x_1$ approach limits $\partial y_i/\partial x_1$ which satisfy the equations

$$(3)\qquad \begin{array}{l}\frac{\partial f_1}{\partial y_1}\frac{\partial y_1}{\partial x_1}+\frac{\partial f_1}{\partial y_2}\frac{\partial y_2}{\partial x_1}+\cdots+\frac{\partial f_1}{\partial y_n}\frac{\partial y_n}{\partial x_1}+\frac{\partial f_1}{\partial x_1}=0,\\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\ \frac{\partial f_n}{\partial y_1}\frac{\partial y_1}{\partial x_1}+\frac{\partial f_n}{\partial y_2}\frac{\partial y_2}{\partial x_1}+\cdots+\frac{\partial f_n}{\partial y_n}\frac{\partial y_n}{\partial x_1}+\frac{\partial f_n}{\partial x_1}=0,\end{array}$$

where the arguments of the derivatives of f are now $(x; y)$. A similar consideration shows the existence of the first derivatives with respect to the variables $x_2, x_3, \cdots, x_m$. The existence of the higher derivatives follows from the observation that the solutions of equations (3) are differentiable $n - 1$ times with respect to the variables x on account of the assumption that the functions f are differentiable n times.

ON A SET OF KERNELS WHOSE DETERMINANTS FORM A STURMIAN SEQUENCE.

BY MR. H. BATEMAN, M.A.

WEYL * has recently given a theorem which states that if a kernel

$$k_n(s, t) = \sum_{p,q=1}^{n} k_{pq}\Phi_p(s)\Phi_q(t) \qquad (k_{pq} = k_{qp})$$

is formed from n functions $\Phi_p(s)$ whose squares are integrable in the interval (0, 1), then the smallest positive root of the

* *Göttinger Nachrichten*, 1911, Heft 2, p. 110.

kernel

$$h_n(s, t) = k(s, t) - k_n(s, t)$$

is not greater than the $(n+1)$th positive root of $k(s, t)$.

It has occurred to me that this theorem is a particular case of the following general theorem:

Let $k(s, t)$ be a symmetric function such that

$$\int_0^1 \int_0^1 [k(s, t)]^2 ds dt$$

is convergent, $a_{pq} = a_{qp}$ $(p, q = 1, 2, \cdots, n)$ a set of constants such that the symmetrical determinant

$$\Delta_n = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

is not zero, $f_1(s)$, $f_2(s)$, $\cdots$, $f_n(s)$ a set of integrable functions such that

$$\int_0^1 [f_p(s)]^2 ds$$

is convergent. Then if

$$h_n(s, t) = \frac{1}{\Delta_n} \begin{vmatrix} k(s, t) & f_1(s) & f_2(s) & \cdots & f_n(s) \\ f_1(t) & a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f_n(t) & a_{n1} & \cdots & \cdots & a_{nn} \end{vmatrix} = \frac{F_n}{\Delta_n}$$

and if $h_{n-1}(s, t)$ is derived from $h_n(s, t)$ by omitting the last row and column in each of the determinants F_n and Δ_n, the roots of the symmetric kernel $h_{n-1}(s, t)$ will be separated by those of $h_n(s, t)$.

Let $D(\lambda)$, $D_{n-1}(\lambda)$, $D_n(\lambda)$ be the determinants of $k(s, t)$, $h_{n-1}(s, t)$, $h_n(s, t)$ respectively; then by a known formula *

$$D_n(\lambda) = \frac{D(\lambda)}{\Delta_n} \nabla_n(\lambda),$$

* *Messenger of Mathematics,* 1908, p. 179.

where

$$\nabla_n(\lambda) = \begin{vmatrix} a_{11} + \lambda\tau_{11} & \cdots & a_{1n} + \lambda\tau_{1n} \\ \cdot & \cdot & \cdot \\ a_{n1} + \lambda\tau_{n1} & \cdots & a_{nn} + \lambda\tau_{nn} \end{vmatrix},$$

$$\tau_{pq} = \tau_{qp} = \int_0^1 f_p(s)\phi_q(s)ds = \int_0^1 \phi_p(s)f_q(s)ds,$$

and $\phi_q(s)$ is the solution of the equation

$$f_q(s) = \phi_q(s) - \lambda\int_0^1 k(s, t)\phi_q(t)dt.$$

Now let A_{pq} denote the cofactor of the constituent $a_{pq} + \lambda\tau_{pq}$ in the determinant $\nabla_n(\lambda)$; then by a property of determinants

$$A_{nn}A_{n-1\cdot n-1} - A^2_{n-1\cdot n} = \nabla_{n-2}(\lambda)\nabla_n(\lambda),$$

where $\nabla_{n-2}(\lambda)$ is derived from $\nabla_n(\lambda)$ by omitting the last two rows and columns.

Now $A_{nn} = \nabla_{n-1}(\lambda)$, hence when $\nabla_{n-1}(\lambda)$ vanishes $\nabla_{n-2}(\lambda)$ and $\nabla_n(\lambda)$ have opposite signs. The functions

$$\nabla_1(\lambda), \quad \nabla_2(\lambda), \quad \cdots, \quad \nabla_n(\lambda)$$

therefore form a Sturmian series and it will be seen presently that the roots of $\nabla_n(\lambda)$ separate those of $\nabla_{n-1}(\lambda)$, the roots of $\nabla_{n-1}(\lambda)$ separate those of $\nabla_{n-2}(\lambda)$, and so on.

Now the roots of $\nabla_n(\lambda) = 0$ are the same as those of $D_n(\lambda) = 0$ and I have shown in a former paper* that the roots of $D_1(\lambda)$ separate those of $D(\lambda)$, hence the functions

$$D(\lambda), \quad D_1(\lambda), \quad D_2(\lambda), \quad \cdots, \quad D_n(\lambda)$$

form a sequence such that the roots of any function in the sequence separate the roots of the preceding function.

If $\overset{+}{\lambda}_1, \overset{+}{\lambda}_2, \cdots, \overset{+}{\lambda}_{n+1}$ are the positive roots of $D(\lambda)$ arranged in order of magnitude, there will be n roots of $D_1(\lambda)$ arranged singly between the gaps, $n - 1$ roots of $D_2(\lambda)$ arranged between the gaps in this second set, and so on. It is clear then that there is at least one root of $D_n(\lambda)$ between $\overset{+}{\lambda}_1$ and $\overset{+}{\lambda}_{n+1}$; this includes Weyl's theorem. We have supposed that the roots of $D(\lambda)$ are all distinct, but the necessary modification for the case of multiple roots is easily introduced.

* *Cambr. Phil. Trans.*, vol. 20 (1908), p. 374. The theorem is used again on p. 182.

If the constants a_{pq} are chosen so that the determinants Δ_n are all positive, $D_{n-2}(\lambda)$ and $D_n(\lambda)$ will have opposite signs when $D_{n-1}(\lambda)$ vanishes, and so the functions

$$D(\lambda), \quad D_1(\lambda), \quad D_2(\lambda), \quad \cdots, \quad D_n(\lambda)$$

will form a Sturmian sequence.

It has been stated that the roots of the functions $\nabla_n(\lambda)$ in the Sturmian sequence separate one another. This is not always true for a Sturmian sequence when the functions are not polynomials, but it can be shown to be true in the present case, as follows. Let $g_n(s)$, $g_n(t)$ be the cofactors of the constituents $f_n(t)$, $f_n(s)$ in the determinant F_n; then from the properties of determinants

$$F_{n-1} \cdot \Delta_n - g_n(s)g_n(t) = F_n \cdot \Delta_{n-1}.$$

Dividing out by $\Delta_{n-1}\Delta_n$, we have

$$h_n(s, t) = h_{n-1}(s, t) - \frac{g_n(s)g_n(t)}{\Delta_{n-1}\Delta_n}.$$

We can now apply the theorem mentioned before to this equation and deduce that the roots of $h_{n-1}(s, t)$ are separated by those of $h_n(s, t)$, there being one root of $h_n(s, t)$ between each consecutive pair of roots of $h_{n-1}(s, t)$.

BRYN MAWR COLLEGE,
November, 1911.

ON THE CUBES OF DETERMINANTS OF THE SECOND, THIRD, AND HIGHER ORDERS.

BY PROFESSOR ROBERT E. MORITZ.

(Read before the San Francisco Section of the American Mathematical Society, April 8, 1911.)

WHILE the square of a determinant of any order may be readily expressed as a determinant of the same order, I am not aware of the existence of a correspondingly simple method by means of which the cube of any determinant may be expressed in determinant form. For a determinant of the fourth order, Δ_4, we have indeed from a well-known property of determinants

$$\Delta_4^3 \equiv \Delta_4',$$

where Δ_4' is the determinant whose constituents are the co-