ITERATED LIMITS OF FUNCTIONS ON AN ABSTRACT RANGE.

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RESERVING details for a later publication, I wish here to indicate a method for the investigation of multiple and iterated limits under general conditions imposed on the range of the independent variable, and free from the unnecessary restrictions involved in the consideration of variables of particular types. For desirable simplicity of the general theory the postulates imposed on the range should not only permit of an effective definition of limiting element, but also be invariant under two fundamental processes:

(*a*) Extension of the system through the adjunction of ideal limiting elements.

(*b*) Composition of systems, e. g., the generation from two linear ranges of a two dimensional range.

In his thesis (Paris, 1906): "Sur quelques points du calcul fonctionnel" (*Rendiconti del Circolo Matematico di Palermo*, volume 22) M. Fréchet supposes the notion of limit of a sequence of elements to be defined for a certain abstract class and secures generalizations of several theorems on point sets and on continuous functions. Later, adopting an analogue of the distance relation (écart) between two points, he secures many more theorems. His postulates, however, do not readily admit of the introduction of ideal elements.

F. Riesz, in a paper before the International Congress of Mathematicians at Rome, 1908 ("Stetigkeitsbegriff und abstracte Mengenlehre," Atti, volume 2 (1908) pages 18–24) proposed a set of postulates in which the fundamental notion is that of a relation between subclass and element of the range, in the sense that the element is a "limiting" element of the subclass. By considering the properties of a system of subclasses that have a common limiting element he is able to define effectively an ideal limiting element as a system of subclasses having certain definite properties. The postulates of Riesz, however, are apparently better fitted for the treatment of abstract point set theory than for the treatment of functions on the range thus characterized.

Let $\mathfrak{P}$ be a class of elements, and let R represent a dyadic relation between subclasses $\mathfrak{R}$ of $\mathfrak{P}$. For the system $(\mathfrak{P}; R)$ we adopt a body of postulates that permit of a very simple definition of limit of a function at an element, in which the classes $\mathfrak{R}$ that have the relation R to a given singular class p (element of $\mathfrak{P}$) play the rôle of "vicinity" or "neighborhood" of a point. We are able to characterize a class u of subclasses $\mathfrak{R}$ as an ideal element in such fashion that the postulates are satisfied by the extended system $(\mathfrak{Q}; S)$, where $\mathfrak{Q}$ is the class $\mathfrak{P}$ with ideal elements adjoined, and S is a relation between subclasses $\mathfrak{S}$ of $\mathfrak{Q}$ defined in terms of R. The extended system $(\mathfrak{Q}; S)$ is closed to this process of extension.

Further, from two such systems, $(\mathfrak{P}'; R')$ and $(\mathfrak{P}''; R'')$, we derive a composite system $(\mathfrak{P}; R)$, where $\mathfrak{P}$ is the product class $\mathfrak{P}'\mathfrak{P}''$ and R is a relation between subclasses of $\mathfrak{P}$ determined by R' and R''. Our postulates are satisfied by the composite system if, and only if, they are satisfied by both component systems. This process of composition may be generalized to give a unique composite system $(\mathfrak{P}; R)$ for any finite number of systems $(\mathfrak{P}^i; R^i)$.

In terms of our relation R we define limit for a sequence of elements so as to fulfil the conditions of the Fréchet limit, and our definition of limiting element of a subclass fulfils the Riesz postulates. We show that in a system satisfying our postulates the definitions of continuity by means of "sequences" and by means of "vicinity" are equivalent. On the other hand our system is less special than the system of Fréchet in which he uses the notion of écart, since a relation R fulfilling our conditions may be defined for any class for which an écart exists. The gain in generality over the Fréchet treatment is seen in the possibility of determining the relation R by means of order relations in which distance and magnitude play no part. An R may be defined for any class having linear order, or for the composite of any finite number of such classes.

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