wishes to be understood in some other sense, which his words do not admit (in other words to *crawl* if we may be permitted to use an expressive vulgarism) can appreciate the labor involved in the production of a text-book with this high record of correctness. We can recommend the book, in spite of some shortcomings, as one from which few readers can fail to get much valuable information without undue effort.

MAXIME BÔCHER.

DIFFERENTIAL INVARIANTS.

Invariants of Quadratic Differential Forms. By J. EDMUND WRIGHT. Cambridge Tract No. 9. Cambridge University Press, 1908. viii + 90 pp.

IN these days, when the number of papers in mathematics published each year is almost without limit and the ramifications are no less perplexing in their variety, one is delighted to find here and there a digest of the work in a particular field. These are the pleasures which the Cambridge Tracts hold in store for us, and we owe a debt of gratitude to our fellow-workers who are willing to pause in their researches to give us a panoramic view of their field and thus to turn over to us in nut-shell form the products of their searchings in the works of their colaborers. The author of the tract before us felt that this was its mission and he seems to have attained his hopes.

In his introduction he leads the reader into the domain of his subject by showing him an invariant and then defining it. The example taken is naturally enough the Gaussian curvature of a surface whose linear element

$$ds^2 = dx^2 + dy^2 + dz^2,$$

expressed in terms of two independent parameters u and v, is written

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2.$$

The idea of differential parameter is also suggested at this time. The reader is then acquainted with the magnitude of the field of inquiry and of the two kinds of problems: (i) The determination of all invariants of one or more differential quadratic forms and their relations; (ii) The geometrical and mechanical interpretation of these invariants.

In Chapter I the author gives a brief historical sketch of the development of the theory, indicating in particular the lines of attack and the leaders in these subfields. Here one finds almost all of the references to the memoirs in which the subject-matter is to be found. Subsequently the reader is inclined to feel that it would have been more helpful had the references been given along with the more or less extended development of the matter as set forth in the succeeding chapters. In these chapters, because of the limited amount of space, the treatment is necessarily brief and so it is desirable that the hurried reader should be able to dip at times into the original memoirs.

Chapter II opens with a short discussion of the problem of finding the necessary and sufficient conditions that two given surfaces be applicable to one another as an introduction to the general problem proposed by Christoffel * which may be stated as follows:

Given two differential quadratic forms in n variables

$$(1) \qquad F = \sum_{i,\kappa} a_{ik} dx_i dx_\kappa, \quad F' = \sum_{i,\kappa} a'_{i\kappa} dy_i dy_\kappa,$$

where $a_{i\kappa}$ and $a'_{i\kappa}$ are functions of the x's and y's respectively; what are the necessary and sufficient conditions that a transformation of the variables exist, say

$$(2) \qquad x_i = \phi_i(y_1, \cdots, y_n),$$

such that F is transformable into F'?

This is equivalent to the determination of the conditions under which the system of partial differential equations

$$(3) \qquad a'_{i\kappa} = \sum_{r,s} a_{rs} \frac{\partial x_r}{\partial y_i} \frac{\partial x_s}{\partial y_\kappa}$$

admits a solution. This question is reduced by Christoffel to an algebraic problem.

If equations (3) be differentiated with respect to y_l, where $l = 1, \cdots, n$, and the resulting equations be solved for the second derivatives, the conditions that these equations be consistent are expressible in the form

$$(4) \qquad (\alpha_1\alpha_2\alpha_3\alpha_4)' = \sum_{i_1 i_2 i_3 i_4} (i_1 i_2 i_3 i_4) \frac{\partial x_{i_1}}{\partial y_{\alpha_1}} \frac{\partial x_{i_2}}{\partial y_{\alpha_2}} \frac{\partial x_{i_3}}{\partial y_{\alpha_3}} \frac{\partial x_{i_4}}{\partial y_{\alpha_4}},$$

* "Ueber die Transformation der homogenen Differentialausdrücke des zweiten Grades," *Crelle*, vol. 70 (1870).

where $(i_1 i_2 i_3 i_4)$ denotes a certain function of the quantities a_{rs} and their first and second derivatives, and $(\alpha_1\alpha_2\alpha_3\alpha_4)'$ a similar function in terms of the functions a'_{rs}. These functions $(i_1 i_2 i_3 i_4)$ were used first by Riemann in the Commentatio Mathematica and so Ricci has called them the *Riemann symbols.*

Similarly to (3) equations (4) are the conditions of equivalence of a form

$$(5) \qquad G_4 = \sum_{i_1 i_2 i_3 i_4} (i_1 i_2 i_3 i_4) d^{(1)}x_{i_1} d^{(2)}x_{i_2} d^{(3)}x_{i_3} d^{(4)}x_{i_4},$$

and an analogous form G'_4 in the y's. Here we take four different sets of differentials, since there are certain linear relations between Riemann symbols, in consequence of which G_4 vanishes identically when the differentials are the same. Thus equations (4) are the conditions for the equivalence of two *quadrilinear* forms.

If equations (4) be differentiated with respect to y_α and if we make use of five index symbols defined by

$$(6) \qquad \begin{aligned} (i_1 i_2 i_3 i_4 i) &= \frac{\partial}{\partial x_i}(i_1 i_2 i_3 i_4) \\ &\quad - \sum_\lambda \left[\left\{ \begin{matrix} i i_1 \\ \lambda \end{matrix} \right\} (\lambda i_2 i_3 i_4) + \left\{ \begin{matrix} i i_2 \\ \lambda \end{matrix} \right\} (i_1 \lambda i_3 i_4) + \cdots \right], \end{aligned}$$

where $\left\{ \begin{smallmatrix} ij \\ k \end{smallmatrix} \right\}$ is the usual Christoffel symbol, we obtain equations

$$(7) \qquad (\alpha_1\alpha_2\alpha_3\alpha_4\alpha)' = \sum_{i_1 i_2 i_3 i_4 i} (i_1 i_2 i_3 i_4 i) \frac{\partial x_{i_1}}{\partial y_{\alpha_1}} \cdots \frac{\partial x_i}{\partial y_\alpha},$$

which evidently are the conditions for the equivalence of two quinquilinear forms G_5 and G'_5.

Continuing by means of a process which is an immediate generalization of (6), we obtain a series of equations analogous to (7) and thus a sequence of covariant forms $G_4, G_5, \cdots, G_\mu$.

The author gives few details in connection with the statement of the foregoing results, but he develops at some length the proof of the following fundamental theorem of Christoffel:

The necessary and sufficient conditions in order that it shall be possible to transform a quadratic form F into another quadratic form F' are that the equations in the variables x, y, $\partial x/\partial y$ derived from the equivalence of two sequences

$$F,\ G_4,\ G_5,\ \cdots,\ G_\mu \quad \text{and} \quad F',\ G'_4,\ \cdots,\ G'_\mu$$

shall be algebraically compatible.

Wright shows that two cases are to be considered:

(i) When an order q can be determined such that the equations

$$F = F', \quad G_4 = G_4', \quad \cdots, \quad G_{q-1} = G_{q-1}'$$

determine the quantities x and $\partial x / \partial y$ as functions of the y's and these values make G_q and G_q' an identity.

(ii) When the above equations yield only $p(< n^2 + n)$ independent equations, and if any set of solutions of these p equations satisfy identically $G_q = G_q'$.

When the conditions of the first case are satisfied the problem is algebraic, but in the second case certain of the quantities left undetermined by the algebraic conditions must satisfy partial differential equations of the first order. From the algebraic theory it follows that the algebraic invariants of the two sets of multilinear forms must be equal. And the equations of transformation are given by equating n independent absolute invariants. Hence it is necessary that the sequence of forms F, G_4, G_5, $\cdots$ be extended to such a point that there shall be n absolute simultaneous invariants.

In particular the invariants I, I_1, $\cdots$ of the algebraic forms F, G_4, $\cdots$ are a complete system of relative differential invariants of the form F, and if under any transformation I becomes κI, then κ is some power of the Jacobian of the transformation.

The remainder of Chapter II is devoted to a discussion of the "absolute differential calculus" which Ricci and Levi-Civita * have developed from the fundamental principles discovered by Christoffel.

Equations (3) may be looked upon as defining a transformation of the functions a_{rs} simultaneously with the transformation (2) of the variables. The same may be said of equations (4) and (7). All of these equations are of the form

$$(8) \qquad Y_{r_1 \ldots r_m} = \sum_{s_1 \ldots s_m} X_{s_1 \ldots s_m} \frac{\partial x_{s_1}}{\partial y_{r_1}} \cdots \frac{\partial x_{s_m}}{\partial y_{r_m}}.$$

Functions $X_{s_1 \ldots s_m}$ whose transformation equations are of this sort are said to form a *covariant system* of order m. In similar manner, the functions $X^{(s_1 \cdots s_m)}$ whose transformation equations

* *Math. Annalen*, vol. 54, pp. 125–201.

are of the form

$$Y^{(r_1 \ldots r_m)} = \sum_{s_1 \ldots s_m} X^{(s_1 \ldots s_m)} \frac{\partial y_{r_1}}{\partial x_{s_1}} \cdots \frac{\partial y_{r_m}}{\partial x_{s_m}}$$

form a *contravariant system* of order m. As an example, the coefficients $a^{(rs)}$ of the quadratic form reciprocal to F form a contravariant system of order two.

Equations (8) are the conditions that the multilinear forms

$$\sum_{r_1 \ldots r_m} Y_{r_1 \ldots r_m} dy_{r_1} \cdots dy_{r_m}, \quad \sum_{s_1 \ldots s_m} X_{s_1 \ldots s_m} dx_{s_1} \cdots dx_{s_m}$$

shall be transformable into one another. They are called *associated forms*.

If $X_{r_1 \ldots r_m}$ and $\Xi_{r_1 \ldots r_m}$ are typical terms of two covariant systems, then $X_{r_1 \ldots r_m} + \Xi_{r_1 \ldots r_m}$ is a term of a covariant system of the same order; in similar manner the quantities $X_{r_1 \ldots r_m} \Xi_{s_1 \ldots s_p}$ form a covariant system of order $m + p$. Analogous results are true of contravariant systems.

Of particular importance is the idea of *composition* of a covariant and a contravariant system. Thus, the two systems whose typical functions are

$$\sum_{s_1 \ldots s_p} \Xi^{(s_1 \ldots s_p)} X_{r_1 \ldots r_m s_1 \ldots s_p} \quad \sum_{s_1 \ldots s_p} \Xi_{s_1 \ldots s_p} X^{(r_1 \ldots r_m s_1 \ldots s_p)}$$

can be shown to be covariant and contravariant respectively of order m.

The particular system $X^{(r_1 \ldots r_m)}$ defined by

$$\sum_{s_1 \ldots s_m} a^{(r_1 s_1)} \cdots a^{(r_m s_m)} X_{s_1 \ldots s_m}$$

is said to be the reciprocal system of $X_{r_1 \ldots r_m}$ with respect to the fundamental form F; and conversely.

There is one other fundamental idea namely *covariant and contravariant differentiation* with respect to a form F. The former is a generalization of equation (6) and the latter is analogous to it. It is significant that if $X_{r_1 \ldots r_m}$ be a term of a covariant system, the functions $X_{r_1 \ldots r_m hk}$ and $X_{r_1 \ldots r_m kh}$, obtained from the first by repeated covariant differentiation with respect to x_h and x_k, and inversely, are not equal in general.

Ricci and Levi-Civita have established by these means the following fundamental theorem :

In order to obtain all the absolute differential invariants of order μ involving the coefficients of F and functions $f_1, \cdots, f_r$ it suffices to determine the algebraic invariants of the following system of forms

(i) $$F, G_4, \cdots, G_{\mu-2};$$

(ii) the associated forms obtained from $f_1, \cdots, f_r$ with respect to F up to order μ.

Wright merely states the general theorem and illustrates the method of proof in the case of a binary form.

In 1884 Lie applied his theory of continuous groups to the determination of the differential invariants of a binary quadratic differential form. This necessitated the introduction of the idea of an "extended" group, consisting of the transformations operating not only upon the original variables x and y, but also upon such added variables, as the coefficients E, F, G of the quadratic form $Edx^2 + 2Fdxdy + Gdy^2$ and certain functions $f_1, \cdots, f_r$ and their derivatives with respect to x and y. This requires a knowledge of the variation of these quantities due to an infinitesimal transformation of the original group, written

$$x' = x + \xi(x, y)\delta t, \quad y = y + \eta(x, y)\delta t,$$

where ξ and η are arbitrary. Then if Ω is the infinitesmal operator of the group in the variables x, y, E, F, G, f, f_x, f_y,

$$\Omega \equiv \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \frac{\delta E}{\delta t}\frac{\partial}{\partial E} + \cdots + \frac{\delta f}{\delta t}\frac{\partial}{\partial f} + \cdots + \frac{\delta f_y}{\delta t}\frac{\partial}{\partial f_y},$$

and an invariant I must satisfy the condition $\Omega I = 0$, whatever be ξ and η. Wright calculates the quantities $\delta E, \cdots, \delta f_y$ directly and gives the complete system of linear equations in the above eight variables which I must satisfy. This is followed by an explanation of the case where the invariants involve functions f and their derivatives of order p, and an exposition of the general method of Forsyth for the calculation of the increments of a function.

The remainder of Chapter III deals with the determination of differential parameters for forms of rank zero, that is forms for which we can find functions u_λ satisfying the condition

(9) $$\sum_{r,s}^{n} a_{rs}dx_s dx_r = \sum_{\lambda}^{n} du_\lambda^2.$$

The variables entering are

(i) $u_1, u_2, \cdots, u_n$ and their total differentials;
(ii) n, or less, arbitrary functions $\phi^{(k)}$ and their derivatives;
(iii) the Jacobian J of the u's.

If we let F_m denote

$$\left\{ \sum_i^n U_i \frac{\partial}{\partial u_i} \right\}^m \phi,$$

where the U's are a set of auxiliary variables, the function F_m is an algebraic form of order m whose coefficients are the mth derivatives of ϕ except for numerical multipliers. It is shown that the functions F are expressible as functions of the quantities a_{rs} and their derivatives. The final result is:

The most general invariant is a function of the quantities $d^r\phi^{(k)}$, the general algebraic invariants of the forms F and of ΣU^2, multiplied by some power of J.

In general more than n functions u_λ are necessary in order that the F may be given the form (9). If there are m ($> n$), then F may be looked upon as a manifold of order n in the space of order m. In this case the problem of finding the invariants of F is that of determining which of the invariants of

$$\sum_1^m du_\lambda^2$$

are invariants of the manifold.

In several memoirs Maschke has developed a symbolic method for differential forms analogous to the Clebsch method for algebraic forms. In Chapter IV Wright gives an exposition of this method. The assumption is made that the quadratic form (1) can be expressed symbolically as $(df)^2$, where f is a symbolic function of the n variables. The derivatives of f, written $\partial f/\partial x_i \equiv f_i$, have no meaning separately, but $f_i f_\kappa$ is to mean $a_{i\kappa}$. As in the case of algebraic forms, equivalent symbols must be introduced in number to the order of the a's appearing in an expression. If $\Phi^{(i)}$, for $i=1, \cdots, n$, are n invariant expressions of the quadratic form F, so that $\Phi^{(i)} = \Phi'^{(i)}$, where $\Phi'^{(i)}$ is $\Phi^{(i)}$ for the new set of variables y, it is shown that the function

$$(10) \qquad |a_{11}a_{22}\cdots a_{nn}|^{-\frac{1}{2}} \left| \frac{\partial \Phi^1}{\partial x_1} \frac{\partial \Phi^2}{\partial x_2} \cdots \frac{\partial \Phi^n}{\partial x_n} \right|$$

is an absolute invariant. This function, denoted by (Φ), is called an *invariantive constituent of the* form F. Any combination of these constituents which has a significance in terms of the a's is an invariant, and thus the invariants are formed.

By (f) is meant the invariantive constituent of equivalent symbols expressing the quadratic form; by (uf) the constituent in which u is any function of the n variables and the remainder of the Φ's in (Φ) are equivalent symbols. In particular, we have

$$(f)^2 = n\,!,\ (uf)^2 = (n-1)\,!\,\Delta_1 u,\ (uf)(vf) = (n-1)\,!\,\Delta(u, v),$$
$$((uf)f) = (n-1)\,!\,\Delta_2 u,$$

where $\Delta_1 u$, $\Delta(u, v)$ and $\Delta_2 u$ are the well-known differential parameters of F. In deriving the last of these identities use is made of covariant differentiation which is expressible thus

$$u^{(i\kappa)} = u_{i\kappa} - \frac{1}{(n-1\,!)} f_{i\kappa}(fa)(ua),$$

f and a being equivalent symbols for the fundamental form, and $u_{i\kappa}$ denoting

$$\frac{\partial^2 u}{\partial x_i \partial x_\kappa}.$$

In the reduction of the invariants to suitable form Maschke makes use of a large number of symbolic identities a few of which Wright gives. The Riemann symbols are expressible in an interesting form, namely

$$(i\kappa rs) = f_{ir} f^{(\kappa s)} - f_{is} f^{(\kappa r)},$$

the subscripts referring to ordinary differentiation of the symbolic function f and the superscripts to covariant differentiation. By means of these symbolic expressions for the convariant forms G_4 and G_5 are obtained. This chapter is very short, and is in fact a resumé with few details of Maschke's memoirs in the fourth volume of the *Transactions*.

The final chapter of the book, which constitutes about half of it, is devoted to applications. The first twelve pages deal with differential parameters of a single form for $n = 2$, the subject matter being similar to that which may be found in Volume III of Darboux's Leçons. The remainder sets forth some of the applications of the absolute differential calculus. The functions used belong to covariant or contravariant systems, and the ordinary operation of differentiation is replaced by covariant or contravariant differentiation with respect to a fundamental form.

The author contributes a solution by these means of the problem of determining the most general triply orthogonal system of surfaces in a three dimensional manifold with a given quadratic form, and arrives at a differential equation of the third order which is an interesting generalization of the equation found by Darboux for euclidean space. The other applications are chosen from those given by Ricci and Levi-Civita. We shall call attention to several of them.

If $\lambda^{(1)}, \cdots, \lambda^{(n)}$ be n functions of $x_1, \cdots, x_n$ satisfying the condition

(11) $$\sum_{ik} a_{ik}\lambda^{(i)}\lambda^{(k)} = 1,$$

the n equations

$$\frac{dx_i}{ds} = \lambda^{(i)}$$

define a family of curves, one through each point, which consequently may be said to be a congruence. It is readily seen that the *coefficients* $\lambda^{(i)}$ form a contravariant system. If λ_i denotes the *reciprocal* function of $\lambda^{(i)}$ then equation (11) may be written in the other forms

$$\sum_{i,k} a^{(ik)}\lambda_i\lambda_k = 1, \quad \sum_r \lambda^{(r)}\lambda_r = 1.$$

If we have n congruences such that the functions $\lambda_{h/r}$ and $\lambda_{k/r}$ (where h and k signalize the congruence and r the derivative) satisfy the conditions

$$\sum_r \lambda_h^{(r)}\lambda_{k/r} = \eta_{kh} \qquad (h, k = 1, \cdots, n),$$

where $\eta_{hk} = 0$ when $h \neq k$, and $\eta_{hh} = 1$, the set of congruences is said to form an *orthogonal ennuple*. An important theorem is that any covariant or contravariant system is expressible in terms of invariants and the coefficients of an orthogonal ennuple. In particular, the first covariant derivatives of the λ's may be expressed in terms of themselves and certain invariants γ_{hkl}, thus

$$\lambda_{h/rs} = \sum_{k,l} \gamma_{hkl}\lambda_{k/r}\lambda_{l/s}.$$

These invariants satisfy the conditions

$$\gamma_{hhq} = 0, \quad \gamma_{hkq} + \gamma_{khq} = 0,$$

and so there are $\frac{1}{2}n^2(n-1)$ of them. They are a generalization of the rotations p, q, r, p_1, q_1, r_1 of the moving axes of Darboux, and like the latter they must satisfy certain partial differential equations of the first order.

The necessary and sufficient condition that a congruence $\lambda_{n|i}$ admit of a family of orthogonal surfaces is

$$\gamma_{nkh} = \gamma_{nhk} \quad (k, h = 1, \cdots, n-1).$$

A geodesic being defined as a curve for which the first variation of the integral of F in (1) is zero, the condition necessary and sufficient that a congruence $\lambda_{n|i}$ consist of geodesics is that $\gamma_{nin} = 0$, for $i = 1, \cdots, n$.

The absolute calculus may be applied also to dynamical problems. Consider any holonomic system and let its generalized coordinates be $x_1, \cdots, x_n$. The kinetic energy of the system, say T, is given by

$$2T = \sum_{r,s} a_{rs}\dot{x}_r\dot{x}_s,$$

where the dot denotes differentiation with respect to the time. If the system is subject to external forces depending only on the position, so that the work done in a small displacement is given by

$$\sum_r X_r dx_r,$$

the equations of motion are those of Lagrange

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_r}\right) - \frac{\partial T}{\partial x_r} = X_r. \tag{12}$$

The given dynamical problem may be regarded as solved when we know the most general values of the x's as functions of t which satisfy (12). These equations define a curve in n-space which is called a *trajectory* of the configuration. By choosing a congruence of these trajectories as one of the systems of an orthogonal ennuple the equations of Lagrange may be given an invariant form of great simplicity. In particular, it may be shown that if the external forces are all zero, the trajectories are the geodesics of the manifold, and they are described with constant velocity. An important result is that the knowledge of an algebraic integral of the system (12) carries with it that of a homogeneous integral of the differential equations for geodesics in the manifold. The chapter closes with a discussion of homogeneous linear and quadratic integrals of the equations of geodesics; the determination in invariant form of the criteria that two dynamical systems have the same trajections; and the geodesic representation of one manifold upon another.

The subject matter is presented in an inspiring way, so that it seems very probable that the reader will turn to the papers of Ricci and Levi-Civita, as the author hopes. The proof reading has been well done and in every way the printer's work is satisfactory.

It is impossible to close this review without remarking the loss to American mathematics by the death of Mr. Wright. His brilliant record at Cambridge and his subsequent career in this country had won for him a high place in his field.

LUTHER PFAHLER EISENHART.

STURM'S GEOMETRISCHE VERWANDTSCHAFTEN.

Die Lehre von den geometrischen Verwandtschaften. Vierter Band: die nichtlinearen und die mehrdeutigen Verwandtschaften zweiter und dritter Stufe. By RUDOLF STURM. Leipzig and Berlin, Teubner, 1909. x + 486 pp.

As the subject matter of this fourth volume of Professor Sturm's extensive treatise on geometric relations is so different from that of the preceding ones,* but little analogy can be drawn with the methods already discussed. With the exception of one elementary treatise, mathematical literature did not include a book on birational transformations before the appearance of the present volume, although over five hundred memoirs have been devoted to the subject during the last two decades. The newness of the subject, the possibility of approaching it from different standpoints, and its applicability to so many other

*These volumes have been reviewed in the BULLETIN; volume 1 in vol. 15, p. 135; volume 2 in vol. 15, p. 252; volume 3 in vol. 16, p. 250.