THE INFINITESIMAL CONTACT TRANSFORMATIONS OF MECHANICS.

BY PROFESSOR EDWARD KASNER.

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1. THE significance of contact transformations in the development of general dynamics and optics, appreciated to some extent by Hamilton, was first brought out explicitly by Lie.* A very thorough and elegant discussion of the whole subject, including a number of new results, has recently been given by Vessiot.† With a conservative dynamical system, defined by its potential energy U (a function of n generalized coordinates) and its kinetic energy T (a quadratic form in the n generalized velocity components), there is associated an infinitesimal contact transformation whose characteristic function W is of special type.‡ The main result of the present note may be stated as follows:

The alternant (Klammerausdruck) of the contact transformations associated with two dynamical systems, of the same number of degrees of freedom, will be a point transformation when, and only when, the expressions for the kinetic energies are either the same or differ merely by a factor.

2. For simplicity and clearness we shall confine ourselves to two degrees of freedom. The infinitesimal contact transformations are then defined by a characteristic function $W(x, y, p)$, each lineal element (x, y, p) being converted into a neighboring element $(x + \delta x, y + \delta y, p + \delta p)$ according to the standard formulas

$$(1) \quad \delta x = W_p \delta t, \quad \delta y = (p W_p - W)\delta t, \quad \delta p = -(W_x + p W_y)\delta t.$$

If the transformation is applied repeatedly to any given element, a series of ∞^1 elements is obtained, the locus of whose points is termed a path curve or trajectory. The direction of the path generated by any element is defined by the formula

* "Die infinitesimalen Berührungstransformationen der Mechanik," *Leipziger Berichte* (1889), pp. 145–153. Lie-Scheffers, Berührungstransformationen, p. 102.

† *Bulletin de la Société Mathématique de France*, vol. 34 (1906), pp. 230–269.

‡ The constant of total energy h is assumed to have a given value, so the discussion is connected with the theory of *natural families*.

$$(2) \qquad m = \frac{\delta y}{\delta x} = p - \frac{W}{W_p}.$$

We shall speak of the direction m as being *transversal* * to the direction p, and shall refer to (2) as the law of transversals connected with the given transformation.

3. In the simplest case of a particle moving in a plane the kinetic energy is of the form $2T = \dot{x}^2 + \dot{y}^2$, and the associated contact transformations are of the type

$$(3) \qquad W = \Omega\,(x, y)\sqrt{1 + p^2}.$$

Lie showed that this type is characterized geometrically by the fact that transversality reduces to orthogonality; or, what is equivalent, each point is converted into a circle of infinitesimal radius with the given point as center. We now prove that *The alternant of any two transformations of type* (3) *is a point transformation.*

For this purpose we make use of the general formula

$$(4) \qquad W_2 = \begin{vmatrix} W_p & W_{1p} \\ W_x + pW_y & W_{1x} + pW_{1y} \end{vmatrix} - \begin{vmatrix} W & W_1 \\ W_y & W_{1y} \end{vmatrix},$$

where W and W_1 are the characteristic functions of any two given transformations and W_2 is the characteristic function of their alternant (commutator, Klammerausdruck). Substituting

$$(5) \qquad W = \Omega\sqrt{1 + p^2}, \quad W_1 = \Omega_1\sqrt{1 + p^2},$$

we find

$$(6) \qquad W_2 = p(\Omega\Omega_{1x} - \Omega_1\Omega_y) - (\Omega\Omega_{1y} - \Omega_1\Omega_y).$$

The linearity of this expression in p proves that it defines a point transformation; its symbol in the usual Lie notation is

$$(6') \qquad (\Omega\Omega_{1x} - \Omega_1\Omega_x)\frac{\partial}{\partial x} + (\Omega\Omega_{1y} - \Omega_1\Omega_y)\frac{\partial}{\partial y}.$$

4. The general case of two degrees of freedom is equivalent to the motion of a particle constrained to remain on an arbitrary surface, whose first quadratic form we write

* We have borrowed this term from the calculus of variations, although the idea seems quite different. The formula (2) however is precisely the transversality formula connected with the problem $\int W dx =$ minimum. There are several other important analogies between the two theories.

$$ds^2 = Edx^2 + 2Fdxdy + Gdy^2. \tag{7}$$

The corresponding contact transformations are

$$W = \Omega\sqrt{E + 2Fp + Gp^2}. \tag{8}$$

The law of transversals becomes

$$E + F(m + p) + Gmp = 0, \tag{9}$$

and expresses orthogonality of directions on the surface (7). Type (8) is also characterized by the fact that each point is converted into a geodesic circle about that point as center. An important special case is dilatation, where the circles are all of equal radii; the characteristic function is then found to be

$$W = \sqrt{\frac{E + 2Fp + Gp^2}{EG - F^2}}. \tag{10}$$

Consider now two transformations

$$W=\Omega\sqrt{E+2Fp+Gp^2}, \quad W_1=\Omega_1\sqrt{E+2Fp+Gp^2}, \tag{11}$$

associated with different potentials on the same surface, and apply formula (4). It is easily verified that W_2 is linear with respect to p. The resulting point transformation is

$$\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y},$$

$$\begin{aligned} \xi &= G(\Omega\Omega_{1x} - \Omega_1\Omega_x) - F(\Omega\Omega_{1y} - \Omega_1\Omega_y), \\ \eta &= E(\Omega\Omega_{1y} - \Omega_1\Omega_y) - F(\Omega\Omega_{1x} - \Omega_1\Omega_x). \end{aligned} \tag{12}$$

Hence if two dynamical systems lead to the same expression for kinetic energy, the alternant of the associated contact transformations is a point transformation.

5. We now inquire whether this can happen when the kinetic energies differ. We may write any two of our transformations in the form

$$W = \sqrt{\alpha + 2\beta p + \gamma p^2}, \quad W_1 = \sqrt{\alpha_1 + 2\beta_1 p + \gamma_1 p^2}, \tag{13}$$

where the coefficients are arbitrary functions of x and y. The value of the alternant W_2 is then a complicated fractional expression. It suffices to observe that the numerator is a polynomial of the third degree in p, and the denominator is WW_1.

For a point transformation it is *necessary* that the expression shall be rational in p. Omitting the trivial case where both W and W_1 are point transformations (which has no dynamical interest), we see that the quadratics under the radicals in (13) can differ only by a factor. The work of § 4 shows that W_2 is then actually a point transformation.

Hence the alternant of the contact transformations (13) *is a point transformation when and only when* $\alpha_1 : \beta_1 : \gamma_1 = \alpha : \beta : \gamma$.

We have now completed the proof of the theorem stated in § 1.

6. The law of transversals for a transformation of the type

(14) $$W = \sqrt{\alpha + 2\beta p + \gamma p^2}$$

is of the form

(15) $$\alpha + \beta(m + p) + \gamma mp = 0.$$

If this is interpreted on a proper auxiliary surface, namely, one whose length element is proportional to $\sqrt{\alpha dx^2 + 2\beta dxdy + \gamma dy^2}$, it expresses orthogonality. In the x, y plane, however, the directions p and m are conjugate with respect to a central conic. The relation (15) is of linear involutorial character, and depends only on the ratios of α, β, γ. We may therefore state the result of § 4 in geometric terms as follows:

If two contact transformations lead to the same linear involutorial law of transversals, their alternant will be a point transformation.

7. We now show that there are no other transversality laws for which an analogous result holds. In the first place we observe from (2) that if two transformations lead to the same law of transversals, the ratio of their characteristic functions is a function of x and y alone. We therefore form the alternant of W and $\lambda(x, y)W$, finding

(16) $$W_2 = W^2\lambda_y - WW_p(\lambda_x + p\lambda_y) = S\lambda_y - \tfrac{1}{2}S_p(\lambda_x + p\lambda_y),$$

where S represents the square of W. The condition that W_2 shall represent a point transformation is found by placing its second derivative with respect to p equal to zero. This gives

(17) $$(\lambda_x + p\lambda_y)S_{ppp} = 0.$$

The first factor vanishes only when λ is constant, a trivial case, since then the two given transformations coincide. The vanishing of the other factor shows that S must be quadratic in p, that is, W must be of the form (14).

Hence two contact transformations with the same transversality law will have a point transformation for alternant only when they are of the type

$$W = \sqrt{\alpha + 2\beta p + \gamma p^2}, \quad W_1 = \lambda \sqrt{\alpha + 2\beta p + \gamma p^2}.$$

Transversality is then expressed by a linear involutorial relation (15), *so that for each point the transversal of a given direction is the conjugate direction with respect to a conic with that point as center.*

8. A less important converse result, relating to the type considered in § 3, we state without proof. The only contact transformations which in combination with *every* transformation of type $W = \Omega \sqrt{1 + p^2}$ give a point transformation for alternant are those of the same type. The same is true even if Ω is restricted to the form $a(x^2 + y^2) + bx + cy + d$, a case of interest since then W converts circles into circles. When a vanishes the transformation belongs to the equilong class of Scheffers.

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ON AN INTEGRAL EQUATION WITH AN ADJOINED CONDITION.

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IN his doctor dissertation * Professor Cairns develops for infinitely many variables the theory of a quadratic form with an associated linear form, in order to prove the existence of solutions of the following integral equation :

$$\phi(s) = \lambda \int_a^b K(s, t)\phi(t)dt + \mu p(s), \tag{1}$$

with the adjoined condition

$$\int_a^b \phi(s)p(s)ds = 0, \tag{2}$$

where $K(s, t)$ is a given continuous symmetric function of s and t, $p(s)$ a given continuous function of s, λ and μ are parameters, and $\phi(s)$ is the function to be determined.

* "Die Anwendung der Integralgleichungen auf die zweite Variation bei isoperimetrischen Problemen," Göttingen, 1907.