HERMITE'S WORKS.

Oeuvres de Charles Hermite. Publiées sous les auspices de l'Académie des Sciences par EMILE PICARD. Vol. II. Paris, Gauthier-Villars, 1908. 8vo. 520 pp.

IN a former number of the BULLETIN * we have given a biographical sketch of Hermite and endeavored to set forth the commanding position he occupied among French mathematicians during the last century. We shall therefore give only a brief notice of the present volume. Its 37 papers bring his publication down to the year 1872 and their chief interest centers about the elliptic functions, the solution of algebraic equations, and the theory of numbers. The reader will perhaps best obtain an idea of the rich contents of this volume if we number its papers in the order in which they occur, arrange them in groups, and indicate briefly their nature. A few of the more important papers will however be analysed at length.

Elementary Algebra.

28) A note of 28 pages written for Gerono and Roguet's Cours, edition of 1856, on homogeneous forms of the second degree in n variables.

Differential Calculus.

37) Elimination of arbitrary functions. Extracted from Hermite's Cours at the Ecole Polytechnique, 1873.

29) Short proof of $1/\rho = d\phi/ds$.

Integral Calculus.

30), 33), 34), 36) Short notes on some integrals, such as

$$\int \frac{x^m dx}{\sqrt{1-x^2}}.$$

Theory of Substitution Groups.

19) Discussion of the nature of $\theta(i)$ in order that

$$\begin{bmatrix} z_i \\ z_{\theta(i)} \end{bmatrix}$$

* 2d ser., vol. 13, p. 182.

may represent a substitution of $z_0, z_1 \cdots, z_{n-1}$. Application to the group of order 168 on 7 letters.

Invariants.

1), 20) Short notes on cubic ternary forms.
9) Reduction of the binary cubic.
10) Resultant of three ternary quadratic forms.
26) The invariants of the binary quintic.
27) The skew invariant of the binary sextic.

Theory of Numbers.

8) A modification of Dirichlet's method of determining the number of classes of binary quadratic forms having a given determinant.

Theory of Functions.

8) Determination of a polynomial $P(x)$ of degree m such that the sum of the squares of the differences between $P(x)$ and a given function for n values of x, $n \geqq m$, shall, when each difference is multiplied by a constant, be a minimum. A different solution from that given by Tchebychef.

21), 22), 23), 24) These four papers occupy 54 pages and constitute one of the most interesting features of this volume. They are a study of different functions of more than one variable which have properties in common with Legendre's function X_n in one variable.

Hermite begins by observing the fundamental nature of the θ's in the elliptic and abelian functions. These θ's are sums of terms of the type $e^{-\phi(x+h, y+h_1, \cdots)}$, ϕ being a quadratic form. The development of this term gives the series

$$\sum \frac{h^n h_1^{n'} \cdots}{n!\; n'! \cdots} U_{n, n' \ldots},$$

where the U's are polynomials in $x, y \cdots$ of degree $n + n' + \cdots$. These U's and a set of associate polynomials V are the first functions which Hermite considered. Their importance in analysis is due, so Hermite thought, partly to the fact that they have many properties of the X_n function and partly to their intimate relation with the abelian functions. After some intermediate steps Hermite arrives at another set of functions which seem to please him better. Let us restrict ourselves to two variables, and set

$$L(a,\ b) = 1 - 2ax - 2by + a^2 + b^2,$$

$$Q(a,\ b) = 1 - 2ax - 2by + a^2(1 - y^2) + 2abxy + b^2(1 - x^2).$$

On developing we get

$$\frac{1}{Q} = \sum a^m b^m U_{m,\,n}, \qquad \frac{1}{L} = \sum a^m b^m V_{m,\,n},$$

where $U_{m,\,n}$, $V_{m,\,n}$ are polynomials in x, y of degree $m + n$. The analogy between $U_{m,\,n}$ and X_n is most perfect. We note that the integral relations analogous to

$$\int_{-1}^{1} X_m X_n dx = 0, \quad \text{etc.}$$

depend on the evaluation of integrals such as

$$\int\int \frac{dxdy}{L(ab)L(a'b')} = \frac{\pi}{ab' - ba'} \arctan \frac{ab' - ba'}{1 - aa' - bb'},$$

$$\int\int \frac{dxdy}{L(ab)Q(a'b')} = \frac{\pi}{aa' + bb'} \log \frac{1}{1 - aa' - bb'},$$

the field of integration being the interior of the unit circle $x^2 + y^2 \leq 1$. We recommend the evaluation of these integrals as an excellent exercise of the reader's ingenuity.

Elementary Elliptic Functions.

13) An elementary presentation of this theory occupying 114 pages. It is taken from a note in Lacroix's Calculus, edition of 1862. This little gem will always have a historical value as coming from the hand of a great master of this theory ; but even to-day one finds points of view which are suggestive and worthy of thought.

14) A transformation of the third order by means of invariants.

17) A method for rapidly computing the coefficients of x^n in the development of *sn*, *cn*, *dn*.

18) Development of $\sqrt[4]{k}$, $\sqrt[4]{k'}$ in series involving q.

31) Development of elliptic integrals of the first and second species.

32) Development of k^2 in powers of $(\omega - i)/(\omega + i)$.
33) Geometric demonstration of the addition theorem.

Application of Elliptic Functions to the Solution of Equations.

2) This is the world-famous solution of the quintic. For three centuries mathematicians had tried in vain to effect its solution.

3) A delightful letter written by the aged Hermite in 1900 to Tannery, giving a proof of the formulas relating to the linear transformation of the modular functions

$$\phi(\omega) = \sqrt[4]{k}, \quad \psi(\omega) = \sqrt[4]{k'}$$

employed in 2). How touching is the close of this letter. "De ma proximitée de l'Espagne je rapporte des cigarettes d'Espagnoles; si vous ne venez pas (after his return to Paris) en fumer avec votre collaborateur d'aujourd'hui, votre professeur d'autrefois, c'est que vous avez le coeur d'un tigre. Totus tuus et toto corde."

4), 5) Application of the invariant theory and the elliptic functions to the solution of the biquadratic. Here is also introduced Hermite's χ function with a table of its linear transformations.

6) Elliptic modular equations. This is a famous memoir, published serially in the *Comptes Rendus* for 1859 and then as a monograph. The point of departure of Hermite's researches on this subject is the actual determination of the resolvents of 7th and 11th degrees of the modular equations $M = 0$ of 8th and 12th degrees, whose existence Galois first pointed out in 1832. The coefficients of all the terms except the absolute term may be found without excessive labor; the last term turns out to be the square root of the discriminant $\sqrt{D}$ of $M = 0$. The calculation of D for equations of so high degree by general methods would be utterly impracticable; it is therefore necessary to make use of the peculiar properties of $M = 0$. Doing this, it is easy to show that D has the form for a prime degree n

$$D = u^{n+1}(1 - u^8)^{n+\epsilon} P^2(u),$$

where u is the function $\phi(\omega)$ above, ϵ the Legendrian symbol $\left(\frac{2}{n}\right)$ and P a polynomial with distinct roots of degree

$$\nu = \frac{n^2 - 1}{8} - \frac{n + \epsilon}{2}$$

in u^8. The determination of the coefficients of P requires now a stroke of genius. Hermite observes that the roots of $P(u)$ correspond to values of u or, what is the same thing, of ω for which $M = 0$ has equal roots. His transformation theory of $\phi(\omega)$ shows him at once that this requires that ω shall satisfy the quadratic equation

$$(1) \qquad P\omega^2 + 2Q\omega + R = 0$$

with negative determinant $-\Delta$ of the type

$$(2) \qquad \Delta = (8\delta - 3n)(n - 2\delta) \text{ or } \Delta = 8\delta(n - 8\delta),$$

where δ is any integer such that $\Delta > 0$. Thus for $n = 7$ and 11, and these are the two cases Hermite is especially interested in as we observed above, we have : $n = 7$, only one determinant, viz., $\Delta = 3$; $n=11$, only two, viz., $\Delta = 7$ and $\Delta = 24$. But conversely, with certain easily determined exceptions, for each of the classes of binary forms corresponding to the determinants (2) there exist two or six quadratic forms (P, Q, R) whose coefficients in (1) give values of ω for which two of the roots of $M = 0$ are equal. Thus with hardly any calculation Herm te finds that for

$$n = 7, \quad P = 1 - u^8 + u^{16}.$$

The case $n = 11$ is more difficult. To determine P in this case Hermite observes that the values of ω given by (1) correspond to complex multiplication.* The resulting modular equations must therefore split up into rational factors. Applying this fact to the modular equation of 12th degree which Sohnke had calculated in 1837, Hermite was able to compute P for this case with relatively little labor.

7) Resolvent of the 7th degree of the modular equation ot the 8th degree.

11) Employs the skew invariant of the quintic to find a quintic resolvent of the general equation of 5th degree.

25) A long memoir of 87 pages devoted to a study of the quintic. The invariant theory is freely used. An especially interesting result is a set of criteria for the reality of the roots.

---

* For a glimpse of this fascinating subject, all too little known in this country, the reader is referred to a review by the author in the BULLETIN, 2d ser., vol. 6 (1900), p. 460.

Application of Elliptic Functions to Quadratic Forms.

12), 15) To make clear to the reader the import of these two papers, in which Hermite's genius shines with such lustre, it will be necessary to go back a little. Abel stated without proof that the equation of transformation of the elliptic functions could be solved by radicals in the case of complex multiplication. This precious fact lay long buried and forgotten until Kronecker resurrected it and made it the point of departure of a long series of brilliant discoveries. He found that the equations in these singular cases split up into rational factors which stand in the most intimate relation with the number of classes of quadratic forms with negative determinants. By this means he found eight fundamental relations which being published (1857–1861) with scarcely any proof were long the wonder and admiration (or envy) of his fellow workers in this field. As an illustration let us cite one of them.

$$F(2m) + 2F(2m - 1^2) + 2F(2m - 2^2) + \cdots = 2\Phi(m),$$

where $F(m)$ is the number of classes of determinant $-m$ in which at least one of the outer coefficients is odd and $\Phi(m)$ is the sum of the divisors of m.

Right well might the mathematical world be astonished when Hermite showed how some at least of them might be obtained by elementary means, using a method already employed by Jacobi. Before explaining how Hermite did this, let us illustrate the method by a simple example which the reader can follow in detail. We know that

$$\frac{4k^2}{\pi^2} = (1 + 2q + 2q^4 + 2q^9 + \cdots)^4 \tag{3}$$

and also that this is equal to

$$1 + 8\sum_n \frac{nq^n}{1 + (-1)^n q^n} = 1 + 8\sum_{m,n} n(-)^{m(n+1)} q^{(m+1)n}. \tag{4}$$

Now the exponent of q in the last member of (4) can take any integral value ν, whereas when we raise the parenthesis to the fourth power in (3) it has the form

$$n_1^4 + n_2^4 + n_3^4 + n_4^4.$$

This gives at once the celebrated theorem of Fermat; every integer can be represented as the sum of four squares. If we consider the coefficient of q^ν in (4) more carefully, we observe that it is eight times the sum of the divisors of ν, if ν is odd; and 24 times the sum of the odd divisors of ν, if ν is even. Thus we get with no difficulty a celebrated theorem due to Eisenstein: the number of representations of the integer ν is eight times the sum of its divisors, when ν is odd; and 24 times the sum of its odd divisor when ν is even. This is the theorem whose proof as given by Eisenstein requires an elaborate knowledge of quaternary quadratic forms.* Let us now consider one of the cases considered by Hermite. Letting $\Theta(z), H(z)$, $\Theta_1(z)$ and $H_1(z)$ be the functions of Jacobi, setting $z = 2Kx/\pi$,

$$\Theta(z) = 1 - 2q\cos 2x + \cdots, \quad \Theta_1(z) = 1 + 2q\cos 2x + \cdots,$$

$$H(z) = 2\sqrt[4]{q}\sin x - 2\sqrt[4]{q^9}\sin 3x + \cdots,$$

$$H_1(z) = 2\sqrt[4]{q}\cos x + 2\sqrt[4]{q^9}\cos 3x + \cdots,$$

Hermite finds

$$(5) \quad \frac{K}{2\pi}\sqrt{\frac{2kK}{\pi}}\,\frac{H^2(z)\Theta_1(z)}{\Theta^2(z)} = A\Theta_1(z) - q\sqrt[4]{q^{-1}}\cos 2x$$
$$- q^4(\sqrt[4]{q^{-1}} + 3\sqrt[4]{q^{-9}})\cos 4x - \cdot$$

On the other hand if we write

$$\frac{H^2(z)\Theta_1(z)}{\Theta^2(z)} = \frac{H(z)\Theta_1(z)}{\Theta(z)} \cdot \frac{H(z)}{\Theta(z)},$$

we have

$$\sqrt{\frac{K}{2\pi}}\,\frac{H(z)\Theta_1(z)}{\Theta(z)} = \sqrt{q}\sin x + \sqrt[4]{q^9}(1 + 2q^{-1})\sin 3x + \cdots,$$

$$\frac{\sqrt{kK}}{\pi}\,\frac{H(z)}{\Theta(z)} = \frac{2\sqrt{q}}{1-q}\sin x + \frac{2\sqrt{q^3}}{1-q^3}\sin 3x + \cdots.$$

If we multiply these two series, we get another expression for the left side of (5). Comparison of these two developments gives

$$A = \sum F(N)q^{\frac{1}{4}N}, \quad N \equiv 3 \mod 4,$$

---

* In our review of vol. 1 we noticed, l. c., p. 185, another demonstration by Hermite of this theorem. Cf. Oeuvres, vol. 1, p. 260.

where $F(N)$ is the number of solutions of

$$N = (2n+1)(2n+4b+3) - 4a^2 \quad (a = 0, \quad \pm 1, \cdots \pm n)$$

and n, b are positive integers. On the other hand Hermite shows that $F(N)$ is nothing but Kronecker's function F defined above. Let us now set $x = 0$ in (5). The left side vanishes, and if we arrange the right side according to powers of q, Hermite finds, letting d', d'' be divisors of N such that $d' > \sqrt{N}$, and $d'' < \sqrt{N}$, that

$$A\Theta(0) = \tfrac{1}{2}\Sigma q^{\frac{1}{4}N}(\Sigma d' - \Sigma d'').$$

The coefficient of $q^{\frac{1}{4}N}$ on the right Kronecker calls $\Psi(N)$; the left side we see is the product of two infinite series in q. Performing the multiplication and equating coefficients of like powers of q gives finally

$$F(N) + 2F(N-2^2) + 2F(N-4^2) + \cdots$$
$$+ 2F(N-4k^2) = \tfrac{1}{2}\Psi(N),$$

a relation between the number of properly primitive quadratic forms with the determinants $-N$, $-(N-4)$, $-(N-16)$, $\cdots$

If we have gone into some details in speaking of the papers 6), 12), and 15), it is partly because their importance demands more than a passing notice and partly with the hope that our remarks may awaken the interest of some reader of this BULLETIN to look farther into these matters.

JAMES PIERPONT.

SHORTER NOTICES.

Serret's Lehrbuch der Differential- und Integralrechnung. Dritte Auflage, dritter Band,* neu bearbeitet von GEORG SCHEFFERS. Leipzig, Teubner, 1909. xii + 658 pp.

THIS book on differential equations is the third and last volume of Scheffer's "Umarbeitung" of the second edition of Serret's Lehrbuch. In comparison with the first two volumes, there are many more alterations made in this third edition of the third volume. In fact one can hardly recognize any traces

*The first two volumes of this work were reviewed in the BULLETIN, vol. 15 (1908–09), p. 140.