mine exactly the place of X if N be even. With an odd number of cards this mode of stacking always brings X to the middle of the pack, since $1 + z = \frac{1}{2}(N + 1)$.

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THE INTEGRAL EQUATION OF THE SECOND KIND, OF VOLTERRA, WITH SINGULAR KERNEL.

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I.

THE integral equation of the second kind, of Volterra, is written

$$(1) \qquad u(x) = \phi(x) + \int_a^x K(x, \xi)u(\xi)d\xi.$$

If the function $K(x, \xi)$ is continuous, $a \leqq \xi \leqq x \leqq b$, and the function $\phi(x)$ is continuous, $a \leqq x \leqq b$, there is one and only one continuous solution of the equation. But if $K(x, \xi)$ is not continuous in its triangular region, the case is more complicated. In I. we consider finite solutions of integral equations of which the kernel $K(x, \xi)$ is absolutely integrable, and after obtaining a theorem for that case apply it to some others where the kernel is no longer absolutely integrable. For this theorem the following conditions limit the given functions of the equation : $K(x, \xi)$ shall satisfy (A) and $\phi(x)$ shall satisfy (B).

(A) A real function of the two real variables x, ξ is to be continuous in the triangle $T: a \leqq \xi \leqq x \leqq b$, $b > a > 0$, except on a finite number of curves each composed of a finite number of continuous pieces with continuously turning tangents. Any vertical portion is to be considered a separate piece, and of such pieces there are to be merely a finite number, $x = \beta_1$, $x = \beta_2$, $\cdots$, $x = \beta_r$. On the other portions of the system of curves there are to be only a finite number of vertical tangents.

(B) In the region $t: a \leqq x \leqq b$ a real function of a single real variable x is to be continuous except at a finite number of points γ_1, γ_2, $\cdots$, γ_s, and is to remain finite.

Let us define the linear region t_δ, formed from t by removing the small portions $\alpha_i - \delta < x < \alpha_i + \delta$ $(i = 1, 2, \cdots, l)$; and the two dimensional region T_δ, formed from T by removing the small strips $\alpha_i - \delta < x < \alpha_i + \delta$ $(i = 1, 2, \cdots, l)$, where the δ is an arbitrarily small magnitude, and the α's, finite in number, are yet to be defined.

Under these conditions we have the

THEOREM. *There is one and only one finite solution of the integral equation* (1), *continuous in t except for a finite number of points, and these points will be among the points $\gamma_1, \cdots, \gamma_s, \alpha_1, \cdots, \alpha_l$ (the α's to be defined below); provided conditions* (A) *and* (B) *and the following further conditions are fulfilled:*

(a) $\int_a^x |K(x, \xi)|\, d\xi$ *converges in t except for a finite number of points $\lambda_1, \cdots, \lambda_t$, and remains finite.*

(b) *There is a finite number of points $\alpha_1, \cdots, \alpha_l$ [including the points β of* (A) *and λ of* (a)] *such that when ϵ and δ are chosen at pleasure there is a length η_δ for which*

$$\int_y^{y+\eta_\delta} |K(x, \xi)|\, d\xi < \epsilon, \qquad (x, y) \text{ and } (x, y + \eta_\delta) \text{ in } T_\delta.$$

(c) *t can be divided into k parts, bounded by points $a = a_0, a_1, \cdots, a_{k-1}, b = a_k$ such that*

$$\int_{a_i}^x |K(x, \xi)|\, d\xi \leqq H < 1 \quad \begin{cases} a_i \leqq x \leqq a_{i+1}, \\ x \neq \alpha_1, \cdots, \alpha_l. \end{cases}$$

In the proof of this theorem (a) and (c) are used in showing the convergence of the expansion of the solution, and (b) in developing what continuity exists.

The condition (A) can be replaced by conditions on the integral of the kernel; for instance (A) and (b) can together be replaced by the condition which follows:

The integral

$$\int_a^x K(x, \xi) r(\xi) d\xi,$$

where $r(x)$ is finite in t and continuous except for a finite number of points, shall converge except at most for a finite number of values of x, and the function of x thus defined shall remain finite; furthermore it shall be continuous except at most for a finite number of values of x, denoted by $\alpha_1, \cdots, \alpha_l$, which are independent of the choice of $r(x)$.

A special case of this theorem has been treated by Mr. W. A. Hurwitz.* The hypotheses for this case were

(B') $\phi(x)$ is continuous in t,

(a') $\int_a^x |K(x, \xi)| d\xi$ converges in t,

(b') $\int_a^x |K(x, \xi)| d\xi$ represents a continuous function in t,

(c') $|K(x_1, \xi)| \geqq |K(x_2, \xi)|$ when $x_1 > x_2$.

Here (a') implies (a), and (b') and (c') together imply (c) and the condition just mentioned that replaces (A) and (b).

By application of the theorem of page 131, with a change of dependent variable, equations of a still more extended type may be solved. In the equations

$$u(x) = \phi(x) + \int_a^x \frac{K(x, \xi)}{f(\xi)} u(\xi) d\xi, \tag{2}$$

$$u(x) = \phi(x) + \int_a^x \frac{K(x, \xi)}{f(\xi) g(x)} u(\xi) d\xi, \tag{3}$$

and

$$u(x) = \phi(x) + \int_a^x \frac{K(x, \xi)}{f(\xi) g(x) \prod_{i=1}^{p} \{ [\xi - \psi_i(x)]^{\lambda_i} \}} u(\xi) d\xi, \qquad \sum_{i=1}^{p} \lambda_i = \lambda < 1, \tag{4}$$

any one of which includes the previous ones as special cases, $K(x, \xi)$ shall satisfy (A) and be finite, $\phi(x)$ shall satisfy (B), $f(x)$ and $g(x)$ shall be continuous in t and unequal to zero except at the point a where they may vanish in any way, and the various ψ's shall be continuous functions of x. Then,

In (2), if $\phi(x) e^{\int_x^b \frac{dx}{|f(x)|}}$ remains finite as x approaches a, there is a solution that vanishes at a as sharply as const. $e^{-\int_x^b \frac{dx}{|f(x)|}}$.

In (3), if $\phi(x) g(x) e^{\int_x^b \frac{dx}{|f(x) g(x)|}}$ remains finite as x approaches a, there is a solution that vanishes at a as sharply as

* As a problem in Professor Bôcher's course in Integral Equations, Harvard University, in 1907–1908.

$$\text{const. } \frac{1}{g(x)} e^{-\int_x^b \frac{dx}{|f(x)g(x)|}}.$$

In (4), if $\phi(x)g(x)e^{\alpha\int_x^b \frac{dx}{[|f(x)g(x)|]^{(\nu+\lambda)/\nu}}}$ remains finite as x approaches a, where α is a certain constant, and ν any constant such that $1 - \lambda > \nu > 0$, there is a solution that vanishes at a as sharply as

$$\text{const. } \frac{1}{g(x)} e^{-\alpha\int_x^b \frac{dx}{[|f(x)g(x)|]^{(\nu+\lambda)/\nu}}}.$$

There may however be more than one finite solution of these equations.

II.

In this section we consider kernels that are not absolutely integrable. We have the following introductory theorem:

Let the kernel of the integral equation (1) be in the form

$$\frac{K(x, \xi)}{G(x, \xi)},$$

where

$G(x, \xi)$ is analytic in T;*

$K(x, \xi)$ is continuous in T, and $\phi(x)$ continuous in t;

$K(x, \xi)$ vanishes at most at a finite number of points in T at which $G(x, \xi)$ also vanishes.

Then there is no solution of (1), continuous in t except for a finite number of points and not identically vanishing through any subinterval of t, unless the kernel $K(x, \xi)/G(x, \xi)$ can be written in the form

$$\frac{\bar{K}(x, \xi)}{g(x)f(\xi)},$$

where $\bar{K}(x, \xi)$ is continuous in T, and $f(x)$ and $g(x)$ are analytic in t.

If $K(x, \xi)/G(x, \xi)$ cannot be rewritten as $\bar{K}(x, \xi)/g(x)f(x)$ for values of ξ, $x_1 < \xi < x_1'$, $x_2 < \xi < x_2'$, $\cdots$, $x_p < \xi < x_p'$, it is necessary for all such values of ξ, if the integral is to converge, that $u(\xi) = 0$. Hence, in general, under such conditions, there

*If we replace the triangle T by the square $S: a \leqq x \leqq b$, $a \leqq \xi \leqq b$, this theorem holds also for the equation with constant limits

$$u(x) = \phi(x) + \int_a^b \frac{K(x, \xi)}{G(x, \xi)} u(\xi) d\xi.$$

will be no solution of the integral equation (1). For that there be a solution under such conditions it is necessary, as is obvious from the form of the equation (1), that the given function $\phi(x)$ satisfy the equations

$$\phi(x) = \int_a^{x_j} \frac{K(x, \xi)}{G(x, \xi)} u(\xi) d\xi, \quad x_j < x < x_j' \quad (j = 1, 2, \cdots, p);$$

wherefore the $\phi(x)$ cannot be chosen arbitrarily in those subintervals of t. The solution when it exists is independent of the value of the kernel in the strips for which $x_j < x < x_j'$, or $x_j < \xi < x_j'$ $(j = 1, 2, \cdots, p)$.

This prepares us to state the

THEOREM. *Let the kernel of* (1) *be in the form*

$$\frac{K(x, \xi)}{f(\xi) g(x)},$$

where

1° (*a*) *$K(x, \xi)$ is continuous in T, and $f(x)$, $g(x)$ and their first derivatives are continuous in t;*

(*b*) *$\partial K(x, \xi)/\partial x$ satisfies* (*A*), *page 130, and is finite in T;*

(*c*) *$\phi(x)$ is continuous in t except at $x = a$ and is such that the function $\phi(x) g(x)$ and its first derivative satisfy* (*B*), *page 130.*

2° *The function $f(x) g(x)$ is greater than zero in the neighborhood of a, and at a vanishes in such a way that*
$$\int_a^x \frac{dx}{f(x) g(x)} \text{ is not convergent;}$$

3° *$\lim_{x=a} [K(x, x) - K(a, a)]/(x - a)^\nu$ exists, where $K(a, a) \neq 0$, and where ν is some number lying between 0 and 1 $(1 > \nu > 0)$ and is also greater than $1 - 1/\{d[f(x) g(x)]/dx\}_{x=a}$;*

4° *$\lim_{x=a} \phi(x) g(x) = 0$.*

Then, under the foregoing conditions,

(*i*) *if $K(a, a) < 0$, there exists one solution of* (1) *continuous in the neighborhood of a and at a, and*

(*ii*) *if $K(a, a) > 0$, there exists a one-parameter family of solutions of* (1) *continuous in the neighborhood of a except perhaps at a itself. As x approaches a, each solution remains less in absolute value than some constant times $f(x)/(x - a)^\nu$.*

If $K(a, a) < 0$ we may take $\nu = 0$ without change in the theorem. Also if $K(a, a) > 0$ and $[df(x)g(x)/dx]_{x=a} < 1$, we may take $\nu = 0$.

A slightly more special theorem, equivalent to taking $\nu = 1$, is obtained by inserting in 1° : *$\partial K(x, \xi)/\partial\xi$ satisfies (A) and is finite in T*, and dropping all of 3° except $K(a, a) \neq 0$.

If we write $K(x, \xi) = K(a, a) + \lambda[K(x, \xi) - K(a, a)]$, the solutions specified in the theorem of page 134 are analytic in the parameter λ, and are the only solutions continuous in the neighborhood of a, except possibly at a, that are analytic in λ. They are also the only solutions continuous in the neighborhood of a, except possibly at a, that satisfy the conditions

$$(a) \qquad \lim_{x=a} \Phi(x) = 0,$$

$$(b) \qquad \Phi'(x) \text{ remains finite,}$$

where

$$\Phi(x) = \int_a^x \frac{K(x, \xi) - K(a, a)}{f(\xi)} u(\xi) d\xi.$$

There are no solutions that satisfy these conditions if 1°, 2°, and 3° of page 134 hold, but not 4°.

III.

So far we have considered only finite solutions, or at most solutions that become infinite at $x = a$ to an order not greater than the first. It is possible, however, to limit the totality of solutions as to character.

THEOREM. *Let the kernel of* (1) *be in the form $K(x, \xi)/f(\xi)g(x)$, where*

1° (*a*) *$K(x, \xi)$ is continuous in T,* and $f(x)$, $g(x)$ and their first derivatives are continuous in t;*

(*b*) *$\partial K(x, \xi)/\partial x$ and $\partial K(x, \xi)/\partial \xi$ satisfy A, and are finite in T;*

(*c*) *$\phi(x)$ is continuous in t except perhaps at a, and is such that the function $\phi(x)g(x)$ and its first derivative satisfy (B);*

2° *The function $f(x)g(x)$ vanishes at most a finite number of times in t;*

3° *On any horizontal line $\xi = \xi_0$ cutting T there is at least one point in T for which $K(x, \xi) \neq 0$.*

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* If we replace T by S this theorem holds for the equation with constant limits.

Then all the solutions of (1) *continuous in* t *except for a finite number of points are such that the function* $u(x)g(x)$ *remains continuous in* t.

This theorem has several applications. If neither $f(x)$ nor $g(x)$ vanishes in t, there can be no solution becoming infinite at any point in t; therefore the continuous solution is the only solution of the equation continuous except at a finite number of points. If $K(a, a) \neq 0$, and if $\lim_{x=a} \phi(x)g(x) = 0$, the theorem of page 134 holds, as we have already noticed. The solutions there given are the only ones continuous except at a finite number of points, such that $d[u(x)g(x)]/dx$ remains finite in t; they are also the only solutions possible, continuous except at a finite number of points, provided that $K(x, \xi) - K(a, a)$ vanishes identically when $\xi = x$.

If the kernel of the integral equation (1)* is analytic in T, and if $\phi(x)$ is continuous in t, a proof similar to that of the above theorem shows that the continuous solution is the only solution of (1) continuous in t except for a finite number of points.

HARVARD UNIVERSITY,
September, 1909.

DESCRIPTIVE GEOMETRY.

Lehrbuch der darstellenden Geometrie für technische Hochschulen, Volume I. By Professor EMIL MÜLLER, of the Imperial Technical School at Vienna. Leipzig and Berlin, Teubner, 1908. xiv + 368 pages, 273 figures, and three plates.

Vorlesungen über darstellende Geometrie. By GINO LORIA. Volume I: *die Darstellungsmethoden*. Authorized German translation from the Italian manuscript, by FRITZ SCHÜTTE. Teubner's Sammlung, volume XXV_1. Leipzig and Berlin, Teubner, 1907. xi + 218 pages and 163 figures.

Descriptive Geometry, a treatise from a mathematical standpoint, together with a collection of exercises and practical applications. By VICTOR T. WILSON, Professor of drawing and design in the Michigan Agricultural College. New York, John Wiley and Sons, 1909. 8vo, viii + 237 pages and 149 figures.

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* If the kernel of the integral equation with constant coefficients is analytic in S and if $\phi(x)$ is continuous in t, the continuous solutions are the only solutions continuous except for a finite number of points.