

tive geometry, but also take part in extensive constructive exercises.

30. Professor Czuber first pointed out that from the result of the investigations of the statistical reports furnished by 60 English and by 28 Austrian insurance companies it is clear that the probability of death is not only a function of the age, but also of the length of the term of insurance. Moreover, other factors must be considered, such as sex, occupation, and personal equations in medical certificates. The problem of the measure of mortality is not so much to determine the functional relation between these numbers, which now seems impossible, as to determine individual cases empirically. The details of this procedure were then illustrated by some numerical examples.

In the general sessions, mention should be made of the paper by Professor Einstein, Bern, "The recent changes which our views of the nature of light have undergone."

In the business meeting of the Vereinigung, which was held on Thursday, September 23, reports of the various committees and officers were read, and appointments for the following year were made. Professors Krause and Schoenflies retired from the executive committee, and their places were filled by the election of Professors E. Czuber and R. Müller. The library and bibliography committee was continued. Professor Rudio reported on the status of the publication of the works of Euler.

Ample provisions were made for social intercourse, two or more entertainments being held every evening. All the participants felt that it was a most successful occasion. The next meeting will be held in Königsberg.

E. DINTZL.

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## GERGONNE'S PILE PROBLEM.

BY DR. H. ONNEN, SR.

IN volume I (1895), page 184, of the BULLETIN, Professor L. E. Dickson has treated Dr. C. T. Hudson's solution of the problem: \* To deal a pack of  $ab$  cards into  $a$  piles of  $b$  cards each and so stack the piles after each deal that after the  $n$ th deal any selected card may be the  $r$ th in the whole pack.

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\* *Educational Times* Reprints, 1868, vol. 9, pp. 89-91. I have tried in vain to lay my hand on Dr. Hudson's article.

Professor Dickson gives the following result. Let  $p_1, p_2, \dots, p_n$  be the places the pile of the selected card is to hold after the first, second,  $\dots$ ,  $n$ th stacking of the piles; then these numbers will be the successive remainders on dividing  $n$  times by any integer  $a$  lying between the limits

$$\frac{a^{n-1}r - b}{b} + \frac{a^n - 1}{a - 1} \quad \text{and} \quad \frac{a^{n-1}(r - 1)}{b} + \frac{a^n - 1}{a - 1}$$

or being equal to one of them.

Moreover Professor Dickson points out that in the particular case

$$p_1 = p_2 = \dots = p_n = \frac{a + 1}{2} \quad \text{and} \quad r = \frac{ab + 1}{2}$$

$n$  stackings are necessary and sufficient, where  $n$  is the least integer for which  $a^{n-1} \geq b$ , and so proves the incorrectness of Dr. Hudson's condition for that case, viz.,  $a^{n+1} + 2 \geq b$ .

As to the general problem the number  $n$  of stackings that are necessary and sufficient may be computed in the following manner:

Since the difference  $v$  between the two limits is

$$v = \frac{a^{n-1}}{b} - 1,$$

the number of stackings  $n$  must be at least so great as to make

$$a^{n-1} \geq b.$$

The least value of all is consequently that which satisfies

$$a^{n-1} \geq b > a^{n-2}.$$

But there does not always exist an integral value for  $n$  between the two stated limits inclusive.

Suppose the lower limit

$$\frac{a^{n-1}(r - 1)}{b} + \frac{a^n - 1}{a - 1} = \delta + \frac{\rho}{b},$$

$\delta$  being an integer and  $\rho < b$ . Then the upper limit will be

$$\delta + \frac{\rho + a^{n-1} - b}{b},$$

and there will be no integer between these limits inclusive if

$$\rho > 0 \quad \text{and} \quad \rho + a^{n-1} - b < b$$

or if

$$0 < \rho < 2b - a^{n-1}.$$

But,  $(a^n - 1)/(a - 1)$  being always an integer,  $\rho$  is the remainder on dividing  $a^{n-1}(r - 1)$  by  $b$ .

Hence we may lay down the following rule for computing the smallest number of stackings that may bring the selected card on the  $r$ th place in the whole pack :

Take  $n$  so that

$$a^{n-1} \geq b > a^{n-2}.$$

Divide  $a^{n-1}(r - 1)$  by  $b$ . If the remainder  $\rho$  be zero or  $\geq 2b - a^{n-1}$ ,  $n$  stackings suffice. But if

$$0 < \rho < 2b - a^{n-1},$$

$n + 1$  stackings are necessary and always sufficient (since  $v = a^n/b - 1 \equiv a - 1$  and  $a$  is at least 2).

In the following I propose to give a further generalization of Hudson's problem.

I suppose the number of piles formed by dealing the cards not to be at every turn the same, but say the first time  $A_1$ , the second time  $A_2$ , etc., the  $n$ th time  $A_n$ . So let the total number of cards  $N$  be

$$N = A_1 b_1 = A_2 b_2 = \dots = A_i b_i \dots = A_n b_n = \dots \quad (1)$$

Again let  $a_1, a_2, \dots, a_i, \dots, a_n$  denote the number of piles placed *beneath* the pile containing the selected card after the first, second,  $\dots$ ,  $i$ th,  $\dots$ ,  $n$ th stacking.

After the first deal the selected card, which for convenience sake may be called  $X$ , is among the  $b_1$  cards of the pile containing it.

At the second deal these  $b_1$  cards are dealt over  $A_2$  piles, so that the pile with  $X$  now contains a smaller group of cards, among which  $X$  must needs be found. And so on. Let  $x_1, x_2, \dots, x_i, \dots, x_n$  denote the group of cards in the pile with  $X$  after the consecutive deals, in which  $X$  must needs be found. Then  $x_1 = b_1$ .

After the  $i$ th deal the pile with  $X$ , counting from the bottom, consists of a group of cards, which certainly does not contain  $X$  and may be denoted by  $m_i$ ; above that the group  $x_i$ , containing  $X$  to a certainty; and finally a group of cards, again certainly without  $X$ .

After the  $i$ th stacking the whole pack, counting from the bottom, consists of  $a_i b_i + m_i$  cards without  $X$ ,  $x_i$  cards containing  $X$ , and again a certain number of cards without  $X$ .

At the following, the  $(i + 1)$ th, deal we have

$$a_i b_i + m_i = m_{i+1} A_{i+1} + r_{i+1}; \quad r_{i+1} + x_i = x_{i+1} A_{i+1} - s_{i+1},$$

$r_{i+1} < A_{i+1}$  and  $s_{i+1} < A_{i+1}$  being the numbers of cards completing respectively the first and the last of the layers formed by the group  $x_i$ .

Putting  $i = 1, 2, \dots, n$  and taking into account that  $m_1 = 0$  and  $x_1 = b_1$ , we get the following two series of equations

After the	<i>Series I.</i>	<i>Series II.</i>
2d deal $a_1 b_1$	$= m_2 A_2 + r_2; r_2 + b_1$	$= x_2 A_2 - s_2,$
3d deal $a_2 b_2 + m_2$	$= m_3 A_3 + r_3; r_3 + x_2$	$= x_3 A_3 - s_3,$
. . . . .	. . . . .	. . . . .
$n$ th deal $a_{n-1} b_{n-1} + m_{n-1}$	$= m_n A_n + r_n; r_n + x_{n-1}$	$= x_n A_n - s_n,$
$n$ th stacking $a_n b_n + m_n$	$= z,$	

$z$  standing for the number of cards beneath the group  $x_n$  after the  $n$ th stacking.

Multiply the two equations representing the situation after the second deal by  $A_1$ , those after the third deal by  $A_1 A_2$ , etc., those after the  $n$ th deal by  $A_1 A_2 \dots A_{n-1}$ , and the equation after the  $n$ th stacking by  $A_1 A_2 \dots A_{n-1} A_n$ . Then adding separately the equations of series I and II and for the sake of brevity putting

$$A_1 = P_1, \quad A_1 A_2 = P_2, \quad \dots, \quad A_1 A_2 \dots A_n = P_n,$$

we get, paying attention to (1),

from I 
$$N \left( a_1 + \sum_{i=2}^n [a_i P_{i-1}] \right) = z P_n + \sum_{i=2}^n [r_i P_{i-1}], \quad (z)$$

from II 
$$N = x_n P_n - \sum_{i=2}^n [(r_i + s_i) P_{i-1}]. \quad (x)$$

If  $x_n = 1$ , the selected card's place in the whole pack is exactly fixed after the  $n$ th stacking.

Now the greatest number of cards with which  $x_n = 1$  may be obtained amounts to  $P_n = A_1 A_2 \dots A_n$ ,  $r_i$  and  $s_i$  being both constantly zero. In this case the equation (z) gives

$$z = a_1 + \sum_{i=2}^n [a_i P_{i-1}]$$

and the place of  $X$  after the  $n$ th stacking becomes

$$1 + z = 1 + a_1 + A_1 a_2 + A_1 A_2 a_3 + \cdots + (A_1 A_2 \cdots A_{n-1}) a_n.$$

This result affords the following variation of the popular trick, which, I believe, is not generally known :

Take a number of cards, being the product of several numbers, for instance  $48 = 4 \times 3 \times 4$ , and deal them successively into as many piles as indicated by the factors, taken in any definite succession, in our example the first time into 4, the second time into 3, and the third time again into 4 piles. After every deal the pile containing the card pitched upon is indicated, and one may stack the piles in any way one pleases, provided the number of piles beneath the pile with the chosen card be at every turn remembered. Suppose this number at the three successive stackings to be 3, 1, and 2 ; then the place of the selected card will be the

$$1 + 3 + 4 \times 1 + 4 \times 3 \times 2 = 32\text{nd.}$$

The calculation may be easily made during the manipulations.

Since  $a_1, a_2, \dots, a_n$  are the remainders on dividing  $z$  successively by  $A_1, A_2, \dots, A_n$ , one may also easily compute the mode of stacking to be performed for shuffling the selected card to any place  $(1 + z)$  fixed beforehand.

If

$$P_n > N > P_{n-1},$$

the  $n$  intended deals into  $A_1, A_2, \dots, A_n$  piles are not always sufficient to fix the place of  $X$  in the pack. This appears from the following three postulates, the correctness of which will be evident by paying close attention to the equations of the series I and II :

*Postulate 1.* If the quantities  $a_i$  are all zero, any  $x_i$  coming from a number  $N < P_n$  can never be greater than the corresponding  $x_i$  originating in the number  $N = P_n$ . Hence for  $a_i = 0$  any number  $N < P_n$  always gives  $x_n = 1$ , since the number  $N = P_n$  furnishes at all events  $x_n = 1$ .

*Postulate 2.* If the quantities  $a_i$  are not all zero, any  $x_i$  may be at most one unit greater than it would be if  $a_i = 0$ . Hence,  $N$  being  $< P_n$ ,  $x_n$  in this case may be 1 or 2.

*Postulate 3.* If  $x_n = 2$  and a  $(n + 1)$ th deal and stacking are to be performed, it may happen that also  $x_{n+1} = 2$ . In this case we have

$$m_{n+1} = m_n, \quad r_{n+1} = s_{n+1} = A_{n+1} - 1,$$

which means that one of the two cards of which  $x_n$  consists is the last of a layer and the other the first of the next layer.

Now considering the particular case

$$A_1 = A_2 \cdots = A_n = A,$$

and consequently

$$b_1 = b_2 = \cdots = l_n = b,$$

as in Dr. Hudson's problem, the equations (z) and (x) become

$$N \sum_{i=1}^n [a_i A^{i-1}] = z A^n + \sum_{i=2}^n [r_i A^{i-1}], \quad (z_1)$$

$$N = x_n A^n - \sum_{i=2}^n [(r_i + s_i) A^{i-1}]. \quad (x_1)$$

Since the limits of  $r_i$  are  $A - 1$  and 0 we have

$$A^n - A \cong \sum_{i=2}^n [r_i A^{i-1}] \cong 0.$$

Carrying these limits into (z<sub>1</sub>) we get

$$\frac{(1+z)A^n - A}{N} \cong \sum_{i=1}^n [a_i A^{i-1}] \cong \frac{zA^n}{N}, \quad (2)$$

*i. e.*, any number

$$\sum_{i=1}^n [a_i A^{i-1}]$$

satisfying these conditions will give such values for  $a_1, a_2, \dots, a_n$  that after the  $n$ th stacking  $z$  cards lie beneath the group  $x_n$  containing the selected card. As however,  $N$  being  $< A^n$ ,  $x_n$  may be 1 or 2 (postulate 2), it will sometimes be uncertain whether  $X$  be the  $(1+z)$ th or the  $(2+z)$ th card.

Putting in Dr. Hudson's limits  $a = A$ ,  $r = 1 + z$  and considering that his quantities  $p_1, p_2, \dots, p_n$  correspond to  $a_1 + 1, a_2 + 1, \dots, a_n + 1$  in my notation, the formula for computing  $a_1, a_2, \dots, a_n$  as proposed by Dr. Hudson may be written thus:

$$\frac{(1+z)A^n - N}{N} \cong \sum_{i=1}^n [a_i A^{i-1}] \cong \frac{zA^n}{N}. \quad (3)$$

Now it appears on further examination that any integer

$$\sum_{i=1}^n [a_i A^{i-1}]$$

satisfying Dr. Hudson's formula (3) gives a mode of stacking the piles, so as to make  $x_n = 1$ . If however such an integer do not exist, there is yet an integer satisfying the somewhat wider limits (2) and affording a proper mode of stacking as to the value of  $z$ ; but with  $x_n = 2$ , so that it remains doubtful whether  $X$  becomes the  $(1 + z)$ th or the  $(2 + z)$ th card, and  $n + 1$  stackings are required to bring it surely to the  $(1 + z)$ th place.

The condition (3) may be written in this manner :

$$\frac{N \sum_{i=1}^n [a_i A^{i-1}]}{A^n} \cong z \cong \frac{N \sum_{i=1}^n [a_i A^{i-1}]}{A^n} - \frac{A^n - N}{A^n},$$

giving one — and only one — value of  $z$ , the mode of stacking being given, if the remainder  $\sigma$  on dividing

$$N \sum_{i=1}^n [a_i A^{i-1}]$$

by  $A^n$  be  $\cong A^n - N$ ; in this case  $x_n = 1$ . If however  $\sigma > A^n - N$ , no integer  $z$  can be found,  $x_n$  being 2.

Taking

$$A_1 = A_2 = \dots = A_n = A$$

and moreover

$$a_1 = a_2 = \dots = a_n = a,$$

so as to put the pile with  $X$  at every turn in the same place between the other piles, the equations (z) and (x) become

$$aN \frac{A^n - 1}{A - 1} = zA^n + \sum_{i=2}^n [r_i A^{i-1}], \tag{z_2}$$

$$N = x_n A^n - \sum_{i=2}^n [(r_i + s_i) A^{i-1}], \tag{x_2}$$

whereas

$$\frac{aN}{A^n} \cdot \frac{A^n - 1}{A - 1} \cong z \cong \frac{aN}{A^n} \cdot \frac{A^n - 1}{A - 1} - \frac{A^n - N}{A^n}.$$

Now an integer may be found for  $z$  if the remainder  $\sigma$  on dividing  $aN \cdot (A^n - 1)/(A - 1)$  by  $A^n$  be  $\cong A^n - N$ . If  $\sigma > A^n - N$ , such an integer does not exist,  $x_n$  being 2.

In the second case one may deal and stack once more, and we have to examine the question whether  $x_{n+1}$  becomes 1 or 2.

In the present case, viz.,  $A_i = A$  and  $a_i = a$ , the numbers  $m_2, m_3, \dots, m_n$  in the equations of series I are increasing till a permanent maximum is attained. This maximum amounts to

$$m_n = m_{n+1} = \frac{ab - r_{n+1}}{A - 1} \quad \text{if } x_n = x_{n+1} = 1$$

or to

$$m_n = m_{n+1} = \frac{ab}{A - 1} - 1 \quad \text{if } x_n = x_{n+1} = 2.$$

Since  $m_n$  and  $m_{n+1}$  are integers,  $ab$  must be a multiple of  $A - 1$  if  $x_n = x_{n+1} = 2$ . Conversely  $x_n = x_{n+1} = 2$  if  $ab$  be a multiple of  $A - 1$ , *except* if  $a = 0$  or  $A - 1$ .

Hence we have the following rule ( $\alpha$ ) for computing whether it is possible or not to fix the place of  $X$  in the pack after any number of stackings, and ( $\beta$ ) if so, whether  $n$  or  $n + 1$  stackings are necessary :

( $\alpha$ ) If  $A - 1 > a > 0$  and  $ab$  a multiple of  $A - 1$ , it is impossible to fix the place of  $X$  precisely. After  $n$  or more stackings its place will be the

$$\left(ab + \frac{ab}{A - 1}\right)\text{th} = \left(\frac{aN}{A - 1}\right)\text{th} \quad \text{or the} \quad \left(1 + \frac{aN}{A - 1}\right)\text{th}.$$

In all other cases the place of  $X$  is determined after  $n$  or  $n + 1$  stackings.

( $\beta$ ) If  $a = 0$ ,  $X$  is the first card after  $n$  stackings. — If  $a = A - 1$ ,  $X$  is the  $N$ th card after  $n$  stackings. — If  $A - 1 > a > 0$  and  $ab$  not a multiple of  $A - 1$ , divide  $aN(A^n - 1)/(A - 1)$  by  $A^n$ . The remainder  $\sigma$  being  $\equiv A^n - N$ , the place of  $X$  is determined after  $n$  stackings. If  $\sigma > A^n - N$ ,  $n + 1$  stackings are required. In both cases  $X$  becomes the

$$\left(1 + \frac{aN - r}{A - 1}\right)\text{th card} \quad (r < A - 1).$$

This number is obviously  $1 +$  the integer of the quotient  $aN/(A - 1)$ .

*Example 1.* Take a pack of 52 cards and make in every deal 4 piles,

$$\therefore A = 4, \quad b = 13.$$



Since

$$4^3 > 52 > 4^2,$$

$n = 3$ . Neither  $a = 1$  nor  $a = 2$  makes  $ab$  a multiple of  $A - 1$ . On dividing  $aN(A^n - 1)/(A - 1)$  by  $A^n$  the remainder is 4 or 8, which is  $< A^n - N = 12$ . After three stackings  $X$  will be:

for  $a = 0$  the 1st card.  
 "  $a = 1$  " 18th "  
 "  $a = 2$  " 35th "  
 "  $a = 3$  " 52d "

*Example 2.* Take a pack of 48 cards. Making at every turn 4 piles, we have

$$4^3 > 48 > 4^2, \quad \therefore n = 3.$$

Now  $ab = 12a$  is always a multiple of  $A - 1 = 3$ , and for  $a = 1$  or  $a = 2$  it is impossible to determine the place of  $X$  exactly, whatever may be the number of stackings. After 3 or more stackings one can declare only that the selected card will be

the 16th or 17th if  $a = 1$   
 " 32d " 33d "  $a = 2$ .

*Example 3.* Suppose  $N = 231$ ,  $A = 7$ ,  $b = 33$ ,  $\therefore n = 3$ .  $ab = 33a$  is divisible by  $A - 1 = 6$  if  $a = 2$  or 4. If  $a = 1$ , 3, or 5, the remainder  $\sigma$  on dividing  $aN(A^n - 1)/(A - 1)$  by  $A^n$  is respectively 133, 56, and 222, the second being  $< A^n - N = 112$ , the two others  $> A^n - N$ . Hence we get

$a = 0$ : after 3 or more stackings  $X$  is the 1st card.  
 $a = 1$ : " 4 " " " "  $X$  " " 39th card.  
 $a = 2$ : " 3 " " " "  $X$  " " 77th or 78th card.  
 $a = 3$ : " 3 " " " "  $X$  " " 116th card.  
 $a = 4$ : " 3 " " " "  $X$  " " 154th or 155th card.  
 $a = 5$ : " 4 " " " "  $X$  " " 193d card.  
 $a = 6$ : " 3 " " " "  $X$  " " 231st card.

If  $A$  be odd and  $a = \frac{1}{2}(A - 1)$ ,  $ab = \frac{1}{2}(A - 1)b$  will be a multiple of  $A - 1$  if  $b$  be even. Hence putting after every deal the pile with  $X$  in the middle, it is impossible to deter-

mine exactly the place of  $X$  if  $N$  be even. With an odd number of cards this mode of stacking always brings  $X$  to the middle of the pack, since  $1 + z = \frac{1}{2}(N + 1)$ .

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THE INTEGRAL EQUATION OF THE SECOND  
KIND, OF VOLTERRA, WITH SINGULAR KERNEL.

BY MR. G. C. EVANS.

(Read before the American Mathematical Society, September 13, 1909.)

I.

THE integral equation of the second kind, of Volterra, is written

$$(1) \quad u(x) = \phi(x) + \int_a^x K(x, \xi)u(\xi)d\xi.$$

If the function  $K(x, \xi)$  is continuous,  $a \leq \xi \leq x \leq b$ , and the function  $\phi(x)$  is continuous,  $a \leq x \leq b$ , there is one and only one continuous solution of the equation. But if  $K(x, \xi)$  is not continuous in its triangular region, the case is more complicated. In I. we consider finite solutions of integral equations of which the kernel  $K(x, \xi)$  is absolutely integrable, and after obtaining a theorem for that case apply it to some others where the kernel is no longer absolutely integrable. For this theorem the following conditions limit the given functions of the equation:  $K(x, \xi)$  shall satisfy (A) and  $\phi(x)$  shall satisfy (B).

(A) A real function of the two real variables  $x, \xi$  is to be continuous in the triangle  $T: a \leq \xi \leq x \leq b, b > a > 0$ , except on a finite number of curves each composed of a finite number of continuous pieces with continuously turning tangents. Any vertical portion is to be considered a separate piece, and of such pieces there are to be merely a finite number,  $x = \beta_1, x = \beta_2, \dots, x = \beta_r$ . On the other portions of the system of curves there are to be only a finite number of vertical tangents.

(B) In the region  $t: a \leq x \leq b$  a real function of a single real variable  $x$  is to be continuous except at a finite number of points  $\gamma_1, \gamma_2, \dots, \gamma_s$ , and is to remain finite.