historical note before the reimpression of the Traité du triangle arithmétique (volume 3, pages 435–444) never mention the name of one of the greatest European mathematicians of the sixteenth century.

GENOA, ITALY,
June, 1909.

DARWIN'S SCIENTIFIC PAPERS.

Scientific Papers. By SIR GEORGE HOWARD DARWIN, K.C.B., F.R.S., Plumian Professor in the University of Cambridge. Vol. I, *Oceanic Tides and Lunar Disturbance of Gravity*, xiv + 463 pp.; Vol. II, *Tidal Friction and Cosmogony*, xvi + 516 pp. Cambridge University Press, 1907, 8. Royal 8vo.

IF one were in need of an example to illustrate the English use of the term "applied mathematics," it would hardly be possible to find a better one than that furnished by the scientific papers of Sir George Darwin. Many investigations ranging from the purest of pure mathematics to the observational portions of the phenomena of nature have at times been placed under this title in our journals and treatises, and after all our language is governed by general usage and not by arbitrary rules. But in the use of an imported term it would seem better, in spite of the customs of late years, to take the foreign value rather than the domestic as the basis of definition. This definition implies an actual or suggested relation between a problem set forth on arbitrary hypotheses and the observed phenomena of matter in space. The course which the argument follows — the laying down of hypothetical laws approximating as nearly as may be to those of nature, the translation into and development of those laws by means of symbols, and the final transference from the symbols back to the phenomena, is well known. The skill of the applied mathematician is chiefly shown in his management of the second of these stages so as to produce as much as possible in the third.

One cannot read any of Professor Darwin's papers without noting this attitude towards his work, and it is markedly shown in the excellent prefaces with which he introduces the papers contained in the two volumes under consideration. In them he gives a clear synopsis of each memoir, with observations on the results obtained. These observations are almost always on the physical aspects of the problems he investigates and but little is said of the mathematical methods and the long and laborious developments which were sometimes necessary to arrive at the ends he had in view. Incidentally, it may be mentioned that it is quite possible to obtain a general idea of the contents of the memoirs from the prefaces alone, a more complete summary being given at the end of each of the longer memoirs. An interesting and important part of the preface to volume II is the author's summary of the present attitude of those best capable of judging towards the more speculative portions of his work.

The papers published by Sir George Darwin up to the present time lie chiefly in the domain of hydrodynamics and elasticity. Part I of the first volume consists mainly of memoirs dealing with the practical problem which has as its final object the prediction of the times and heights of the tides at any port. There are two general methods of attacking this question. The first, known as the equilibrium theory, is based on the supposition that the position which the water on the earth has at any instant is the same as that which it would have if the centers of the earth and moon were for that instant in their actual positions but relatively at rest. In other words, in the equations of motion we neglect all the effective forces (mass accelerations), except that due to the rotation of the earth round its axis. Even with this simplification the application of the theory to actual tidal effects is still remote, on account of the considerable effect which the distribution of land and water has on the height and time of the tide at any place. The "corrected equilibrium theory" attempts to take some account of this distribution by the evaluation of certain integrals over the water boundaries. This matter is dealt with numerically by Darwin in conjunction with H. H. Turner in paper 8. The boundaries in general are supposed to follow lines of latitude and longitude, $l =$ const., $\lambda =$ const.; Professor Turner has obtained a closer approximation by making some of them follow the lines $\pm al \pm b\lambda =$ const., where a, b are small integers. It would be interesting to compare these results with integrals computed by following Love's harmonic division of the earth's surface.*

The only practical use which is made of the dynamical theory is the discovery of the periods of the principal tides on the sup-

* Presidential address, Section A, B. A. A. S. 1907.

position that the times and heights of the water at any place may be expressed as a sum of harmonic terms, each of which constitutes a "tide." Theoretically, the amplitude and phase of each tide are also determinate but owing to mathematical difficulties, they are always obtained from the observations. The number of terms to be taken into account is of course infinite, but for most places only ten or twelve need be considered, the observations themselves showing that the sum of the remainder may be neglected. Sir George Darwin has succeeded in putting this problem into an economical form. First, the determination of the constants, the same for all time for a particular port, from the observations of the tides for a year or more; and second, the approximate determination of the tide, when these constants and the position of the moon are given, by methods which are within the grasp of the ordinary navigator. More than half the first volume is devoted to these questions.

An interesting point in connection with the terms or tides of long period (a fortnight or a month) is the possibility that they would be considerably affected by an elastic yielding of the solid earth to the tide-generating forces. This was subjected to computation by Sir George Darwin and thence he deduced evidence that the earth's effective rigidity is at least as great as that of steel. The volume closes with the papers which contain an account of the experiments performed by himself and Horace Darwin to measure the lunar disturbance of gravity. This attempt did not meet with success at the time, but quite lately it has been found possible by means of improved apparatus to observe this disturbance.

It is in the second volume that we find the memoirs in which Sir George Darwin has developed the evolution theory of tidal friction which has its chief application in the past history of the earth and moon. In tracing this history back, several stages are to be noted. The first or present stage is that in which we consider an earth practically rigid and more or less covered by water. The attractions of the moon and sun cause tides which give rise to secular effects on account of friction in the motion of the water relative to the earth. The effect is two-fold: it increases the period of rotation of the earth on its axis and also the period of rotation of the moon round the earth. The changes in these two periods are not the same, and the combined effect to an observer situated on the earth is an apparent shortening of the period of rotation of the moon round the earth. Some other body must be observed in order to separate the two effects: Professor Newcomb had hoped to examine the observations of Jupiter's satellites for this purpose. Since it directly concerns our measure of time, the question is of great importance and it may be advisable to collect observations of every kind which can be affected and examine them with this end in view.

A certain disagreement between observation and gravitational theory in the mean period of the moon's motion has long been known under the name of the "unexplained secular acceleration of the moon's mean motion" and it has most generally been ascribed to this differential effect of tidal friction. For many years its amount was considered to be such as to change the mean longitude by six seconds of arc in one century from the epoch of reckoning. Later, Newcomb's investigations changed it to four and later still to about two seconds. More careful research into the records of ancient eclipses, which form the chief means for obtaining the number, by Newcomb, Cowell, and others have shown how little dependence can be placed on the majority of them, and the number of eclipses to be rejected for this reason has continually increased. It is now doubtful whether we can state that there is any residual effect greater than one second to be ascribed to tidal friction or other unknown cause. This, of course, does not invalidate Darwin's work, since the cause and therefore the effect, though small, is real; it simply alters the time estimates for the total changes. Other effects are changes in the inclination and eccentricity of the lunar orbit.

The second stage is that of a viscous or imperfectly elastic earth in which the main tides are bodily, and it is in this stage that the past history of the earth and moon has been most fully developed by the author. The results are too well known to need restatement here. Volume II contains the memoirs in which they were developed, the paper embodying his remarkable discussion of the series of changes in the orbit of the moon due to the tidal distortion of the earth being that numbered 6. Looking backward in time, we observe the moon brought comparatively near to the earth.

In the third stage the earth is treated as a liquid body in which the effects of viscosity are probably small compared with those of inertia, in contradistinction to the second stage for which the reverse is the case. The problem is now that of a rotating liquid mass disturbed by a body moving round it and at a distance which may be nearly of the same order of magnitude as the tidal distortion. In the fourth, and in this development the final, stage the two bodies form one liquid rotating mass; the separation is supposed to have been effected by the period of rotation being too short for secular stability. The problem is therefore a consideration of the forms and stabilities of the equilibrum of rotating liquid masses under gravitation. The memoirs dealing with these stages will be given in volume III.

We have then a series of problems the connection between which is continuous from the physical point of view, but for which the mathematical representation is different. These differences of representation may be due to the mathematical difficulties which are presented in an attempt to treat them all as successive stages arising from giving to the time successive values in the same set of formulas, but more probably they are due to the real impossibility of obtaining a single formula adapted to numerical computation for the whole series of changes. We have an analogy in the representation of an analytic function containing singular points in the finite part of the plane; the same power series will not serve for every region of the plane. To continue the analogy, we may say that the different regions for which the formulas are available have been to a certain extent mapped out and the formulas more or less accurately obtained, but the connections between these regions are narrow and the labor of finding values for the function in these connecting portions, so as to make the numerical results continuous, becomes very great. Thus the gaps in Darwin's work which are most serious from the point of view of evolution are those which occur between the successive stages. But though mathematical analysis has not succeeded in bridging them, the crossing is less difficult, if less accurate, by argument from general physical principles.

The method for the solution of these problems adopted by Sir George Darwin is nearly always based on that used to solve the equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

When the body approximates to a sphere the potentials are expanded in a series of spherical harmonics and these harmonics, in order to satisfy the hydrodynamical equations, have coefficients which depend on the time. Much careful consideration is necessary in making the choice of those harmonics which will produce sensible effects. And the time element is an important factor. If the harmonics are periodic with respect to the time, we obtain oscillatory changes the magnitudes of which are directly seen, and they produce no secular changes in the system. But if they are not periodic, the changes produced will vary very considerably with the length of the interval over which they are discussed. We cannot then neglect non-periodic terms even when the coefficients may be small over intervals of time comparable with those of the periods of the larger terms whenever it is desired to discuss secular changes. This is one of the chief pitfalls in many of the problems of celestial mechanics. The necessity for using infinite series of functions the coefficients of which do not follow any simple law carries with it the necessity for stopping the series at some point, and in the great majority of cases we are not able to discover the error committed by so doing. Thus a doubt will continually arise as to whether some neglected term or terms may not considerably modify the results over the chosen interval of time. The doubts may become greater when the constants of the solution have to be determined from observations which should extend far back into the past to obtain the accuracy demanded.

As will be gathered from the above remarks, the arrangement of the papers is mainly by subjects, so that each volume is to a certain extent complete in itself. A chronological list of the papers published to date is given. The third volume containing papers on figures of equilibrium of rotating liquids will embody the investigations which lead to the possibility that the moon separated off from the earth by fission; while the fourth volume will consist of the memoir on periodic orbits, addresses, and miscellaneous papers not included under the preceding titles.

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YALE UNIVERSITY,
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