dimensions is obtained. A subgroup of G_{n^2-1} is obtained when the elements of M satisfy certain conditions, as *e. g.*, the well-known conditions defining the orthogonal group. Professor Newson's fundamental theorem lays down the necessary and sufficient conditions which its elements of M must satisfy in order to have a subgroup of G_{n^2-1}. He defines a complete family of automorphic forms ϕ_i which are homogeneous and symmetric functions in from 1 to n sets of n variables each. His theorem is: A necessary and sufficient condition for the existence of a subgroup of G_{n^2-1} is that the elements of M satisfy a set of equations $\phi_i = l_i$ consisting of a complete family of automorphic forms in the elements of the rows or columns of M, each equated to the corresponding coefficient of the family.

Families of lower degrees define continuous subgroups of G_{n^2-1}; after a certain degree is reached the subgroups become discontinuous; above a certain other degree the conditions are satisfied only by the identical transformation. The paper will be published in the *Kansas University Science Bulletin.*

32. Mr. Schweitzer contrasted the formal properties of Bolzano's linear series with his exposition of the series of Vailati (the system ${}^1R_1$) and showed how to extend Bolzano's series to n dimensions ($n = 1, 2, 3, \cdots$) by considering simple modifications of the axioms of dimensionality and extension in his system ${}^nR_n$. Application of the author's n-dimensional open and closed chains is made.

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THE GROUPS WHICH MAY BE GENERATED BY TWO OPERATORS s_1, s_2 SATISFYING THE EQUATION $(s_1s_2)^\alpha = (s_2s_1)^\beta$, α AND β BEING RELATIVELY PRIME.

BY PROFESSOR G. A. MILLER.

(Read before the American Mathematical Society, September 13, 1909.)

Since s_1s_2 and s_2s_1 are of the same order and α, β are relatively prime, it results that this common order is prime to both α and β. Hence s_1s_2 and s_2s_1 are generated by either $(s_1s_2)^\alpha$ or $(s_2s_1)^\beta$, and the cyclic group generated by s_1s_2 coincides with the one generated by s_2s_1. A direct consequence of this is that the group generated by s_1s_2 is invariant under the entire group G

generated by s_1, s_2. It is also evident that G is generated by s_1s_2, s_1, and that every group which involves an invariant cyclic subgroup and is generated by this subgroup and an additional operator may also be generated by two operators satisfying the conditions imposed upon s_1, s_2 in the heading of this paper. These facts establish the theorem:

The totality of the groups which may be generated by two operators satisfying the condition $(s_1s_2)^\alpha = (s_2s_1)^\beta$, *$\alpha$ and β being relatively prime, coincides with the totality of those generated by two operators which are such that one of them is transformed into a power of itself by the other.*

The groups which may be generated by two operators t_1, t_2 having a common square have been called the generalized dihedral rotation groups.* That these groups are included in the category satisfying the conditions imposed in the heading of the present paper results directly from the facts that

$$t_1t_2^{-1} = (t_2^{-1}t_1)^{-1}, \quad \{t_1, t_2\} \equiv \{t_1, t_2^{-1}\}.$$

Hence this category of groups may be regarded as composed of those groups which result from a second generalization of the dihedral rotation groups; the first generalization corresponds to the special values $\alpha = 1$, $\beta = -1$. As $(s_1s_2)^\alpha = (s_2s_1)^\beta$ expresses only one condition between s_1, s_2 this equation must be satisfied by the generators of an infinite number of distinct groups of finite order for every pair of values of α, β.†

One of the most interesting features of the relation $(s_1s_2)^\alpha = (s_2s_1)^\beta$ is that it may be used as one of the two conditions which are satisfied by the two generators of only a finite number of groups. This fact may be established as follows: If $s_1^a = s_2^b$, it is evident that s_1^a is invariant under G and hence $\{s_1^a, s_1s_2\}$ is an abelian group. When a and b are relatively prime this abelian group coincides with G; for otherwise the quotient group of G with respect to this abelian group would have an order prime to b, and s_2 would correspond to an operator differing from the identity in this quotient group. This is impossible since s_2^b would have to correspond to identity. Having proved that G is abelian whenever α, β and a, b are two pairs of relatively prime numbers, it is not difficult to prove that there is only a finite number of distinct operators which can satisfy both of these conditions for given values of

* *Archiv der Mathematik und Physik*, vol. 9 (1905), p. 6.

† *Amer. Jour. of Mathematics*, vol. 31 (1909), p. 167.

the relatively prime pairs α, β and a, b. We shall prove this fact in the following paragraph.

Since G is abelian $(s_1s_2)^\alpha = (s_2s_1)^\beta = (s_1s_2)^\beta$, and hence $(s_1s_2)^{\alpha-\beta} = s_1^{\alpha-\beta}\, s_2^{\alpha-\beta} = 1$, or $s_1^{\alpha-\beta} = s_2^{\beta-\alpha}$. Combining this equation with $s_1^a = s_2^b$, there results $s_1^{a(\alpha-\beta)} = s_2^{a(\beta-\alpha)} = s_2^{b(\alpha-\beta)}$, and hence

$$s_2^{(a+b)(\beta-\alpha)} = 1 = s_1^{(a+b)(\alpha-\beta)}.$$

As the orders of s_1, s_2 are limited and these operators must be commutative, this proves that *only a finite number of groups can be generated by two operators which satisfy both of the equations*

$$(s_1s_2)^\alpha = (s_2s_1)^\beta \quad \text{and} \quad s_1^a = s_2^b,$$

where α, β and a, b represent two pairs of relatively prime numbers. For instance, when these numbers are 4, 5 and 2, 3 G is the group of order 5. That is, if s_1, s_2 satisfy both of the equations

$$(s_1s_2)^4 = (s_2s_1)^5,\ s_1^2 = s_2^3,$$

they must generate the group of order 5. This result establishes close contact between the present note and the paper "On groups which may be defined by two operators satisfying two conditions," *American Journal of Mathematics,* volume 31 (1909), page 167.

A NOTE ON IMAGINARY INTERSECTIONS.

BY PROFESSOR ELLERY W. DAVIS.

In the plane let there be a conic C and a line L. Set up a system of coordinates such that L is the line infinity, its pole O with regard to C is the origin, the axes OX and OY are conjugate with regard to C, while X and Y are their intersections with L. Furthermore let $x = \pm 1$, $y = \pm 1$ be tangents to C through Y and X respectively. Then $x =$ a constant passes through Y, while $y =$ a constant passes through x. All these lines are to be determined by the fact that any four convergents form a harmonic set when the constants in the right member are a harmonic set of numbers. In brief, C, L, and the coordinates are projectively transformed from a circle $x^2 + y^2 = 1$, the line infinity, and a rectangular system whose origin is the center of the circle. The equation of any line in the transformed coordinates is precisely the same as that of which it is the projection in the rectangular coordinates.