The norm of the number on the left is found to be p. It seems impracticable to determine whether or not p has actual prime factors in the field of 128th roots of 1, but this is very improbable, as the class number in that field is a multiple of 21,121.*

The use of complex numbers appears to be of no assistance in the problem of determining whether F_n is prime or composite.

AN EXTENSION OF CERTAIN INTEGRABILITY CONDITIONS.

BY PROFESSOR J. EDMUND WRIGHT.

SUPPOSE there are n functions $a_1, a_2, \cdots, a_n$ of n independent variables $x_1, x_2, \cdots, x_n$, satisfying the conditions

$$\frac{\partial a_p}{\partial x_q} - \frac{\partial a_q}{\partial x_p} = 0$$

for all values of p and q. It is well known that the functions a must all be first derivatives of a single function V. Similarly, if there are $\frac{1}{2}n(n+1)$ functions a_{pr} such that $a_{pr} = a_{qp}$, satisfying the relations

$$\frac{\partial a_{pq}}{\partial x_r} = \frac{\partial a_{pr}}{\partial x_q}$$

for all values of p, q, r, then the a's must be second derivatives of a single function.

The following question arises in connection with an application of the theory of invariants of quadratic differential forms: Suppose there are $n(n+1)$ functions H_{pq}, K_{pq} such that $H_{pq} = H_{qp}$, $K_{pq} = K_{qp}$, satisfying the conditions

$$\frac{\partial}{\partial x_r}(H_{pq}) + K_{pq}\frac{\partial Y}{\partial x_r} = \frac{\partial}{\partial x_p}(H_{qr}) + K_{qr}\frac{\partial Y}{\partial x_p},$$

for all values of p, q, r; Y being a given function of the variables; what are the conditions on the functions H, K?

We first consider the case of $2n$ functions $a_1, a_2, \cdots, a_n$; $b_1, b_2, \cdots, b_n$, satisfying the conditions

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* Reuschle, Tafeln, p. 461.

$$(1) \qquad \frac{\partial a_p}{\partial x_q} - \frac{\partial a_q}{\partial x_p} = b_q \frac{\partial Y}{\partial x_p} - b_p \frac{\partial Y}{\partial x_q}.$$

Take three equations of the type (1), those for (p, q), (q, r), (r, p), differentiate the first with respect to r, the second with respect to p, the third with respect to q, and add. The quantities a are eliminated, and we have the result

$$(2) \qquad (b_{qr} - b_{rq})Y_p + (b_{rp} - b_{pr})Y_q + (b_{pq} - b_{qp})Y_r = 0,$$

where additional suffixes denote differentiation.

Now the equations (1) are unaltered if we replace b_p by $b'_p + \lambda Y$, where λ is an arbitrary function of the variables, and functions b and λ can be determined to satisfy the two equations

$$b_1 = \lambda Y_1 + \frac{\partial b}{\partial x_1}, \qquad b_2 = \lambda Y_2 + \frac{\partial b}{\partial x_2},$$

for elimination of λ gives a single equation for b, and any solution of this, combined with one of the above equations serves to determine λ.

We may thus in equations (1), (2), assume b_p replaced by b'_p, where b'_1 and b'_2 are first derivatives of a function b. Also we write

$$b'_p - \frac{\partial b}{\partial x_p} = b''_p.$$

In equation (2) give p, q, r, the values 1, 2, 3. It becomes precisely

$$J\begin{pmatrix} \lambda, & b''_3 \\ x_1, & x_2 \end{pmatrix} = 0,$$

and therefore b''_3 is a function of Y, x_3, x_4, $\cdots$, x_n only.

We can therefore find a function $F(Y, x_3, x_4, \cdots, x_n)$ such that $b''_3 = (\partial F/\partial x_3)_0$ where the suffix indicates that Y is kept constant. Hence

$$b''_3 = \frac{\partial F}{\partial x_3} - \frac{\partial F}{\partial Y} Y_3;$$

also

$$\frac{\partial F}{\partial x_1} = \frac{\partial F}{\partial Y} Y_1, \qquad \frac{\partial F}{\partial x_2} = \frac{\partial F}{\partial Y} Y_2,$$

and therefore

$$b_1=\left(\lambda-\frac{\partial F}{\partial Y}\right)Y_1+\frac{\partial}{\partial x_1}(b+F),$$

$$b_2=\left(\lambda-\frac{\partial F}{\partial Y}\right)Y_2+\frac{\partial}{\partial x_2}(b+F),$$

$$b_3=\left(\lambda-\frac{\partial F}{\partial Y}\right)Y_3+\frac{\partial}{\partial x_3}(b+F),$$

or, changing the notation, we have found functions b and λ such that $b_p=\lambda Y_p+b'_p$, and $b'_p=\partial b/\partial x_p$ for $p=1, 2, 3$.

If we now apply (2) for the three sets of values (1, 2, 4), (2, 3, 4), (3, 1, 4), we get

$$J\begin{pmatrix} Y, b''_4 \\ x_1, x_2 \end{pmatrix}=0,\quad J\begin{pmatrix} Y, b''_4 \\ x_2, x_3 \end{pmatrix}=0,\quad J\begin{pmatrix} Y, b''_4 \\ x_3, x_1 \end{pmatrix}=0,$$

and hence b''_4 is a function of $Y, x_4, x_5, \cdots, x_n$ only. As before we may modify λ and b, so as to make $b'_p=\partial b/\partial x_4$ for $p=1, 2, 3, 4$, and the process may be continued so that finally we have

$$b_p=\lambda Y_p+\frac{\partial b}{\partial x_p} \tag{3}$$

for all values of p.

Again, from (1),

$$\frac{\partial a_p}{\partial x_q}-\frac{\partial a_q}{\partial x_p}=\frac{\partial b}{\partial x_p}\frac{\partial Y}{\partial x_q}-\frac{\partial b}{\partial x_q}\frac{\partial Y}{\partial x_p}$$

$$=\frac{\partial}{\partial x_p}\left(b\frac{\partial Y}{\partial x_q}\right)-\frac{\partial}{\partial x_q}\left(b\frac{\partial Y}{\partial x_p}\right),$$

or

$$\frac{\partial}{\partial x_q}\left(a_p+b\frac{\partial Y}{\partial x_p}\right)=\frac{\partial}{\partial x_p}\left(a_q+\frac{\partial Y}{\partial x_q}\right),$$

and therefore

$$a_p=-b\frac{\partial Y}{\partial x_p}+\frac{\partial Z}{\partial x_p},$$

where Z is a new function. The complete solution of (1) is therefore given by

$$a_p=-b\frac{\partial Y}{\partial x_p}+\frac{\partial Z}{\partial x_p},\quad b_p=\lambda\frac{\partial Y}{\partial x_p}+\frac{\partial b}{\partial x_p}, \tag{4}$$

where Z, b, λ, are three arbitrary functions.

Now consider the equation

$$(5) \qquad \frac{\partial}{\partial x_r}(H_{pq}) + K_{pq}\frac{\partial Y}{\partial x_r} = \frac{\partial}{\partial x_q}(H_{pr}) + K_{pr}\frac{\partial Y}{\partial x_q}.$$

Keep p fixed, and let $H_{pq} = a_q$, $K_{pq} = -b_q$. We now have equation (1), and hence

$$(6) \qquad H_{pq} = -B_p\frac{\partial Y}{\partial x_q} + \frac{\partial Z_p}{\partial x_q},$$

$$(7) \qquad K_{pq} = \lambda_p\frac{\partial Y}{\partial x_q} + \frac{\partial B_p}{\partial x_q},$$

where λ_p, B_p, Z_p, denote $3n$ as yet arbitrary functions.

Again, $H_{pq} = H_{qp}$, and therefore from (6)

$$\frac{\partial Z_p}{\partial x_q} - \frac{\partial Z_q}{\partial x_p} = B_p\frac{\partial Y}{\partial x_q} - B_q\frac{\partial Y}{\partial x_p}.$$

This equation is of the same type as (1), and hence

$$(8) \qquad Z_p = -B\frac{\partial Y}{\partial x_p} + \frac{\partial C}{\partial x_p}, \quad B_p = \nu\frac{\partial Y}{\partial x_p} + \frac{\partial B}{\partial x_p}.$$

Similarly from the condition $K_{pq} = K_{qp}$ we have the equations

$$(9) \qquad B_p = -\lambda\frac{\partial Y}{\partial x_p} + \frac{\partial \eta}{\partial x_p}, \quad -\lambda_p = -\mu\frac{\partial Y}{\partial x_p} + \frac{\partial \lambda}{\partial x_p}.$$

It follows without difficulty that $\nu = -\lambda$, $\eta = B$, and hence, substituting in (6) and (7) we have the final results

$$H_{pq} = \lambda\frac{\partial Y}{\partial x_p}\frac{\partial Y}{\partial x_q} - \frac{\partial B}{\partial x_p}\frac{\partial Y}{\partial x_q} - \frac{\partial B}{\partial x_p}\frac{\partial Y}{\partial x_q} - B\frac{\partial^2 Y}{\partial x_p \partial x_q} + \frac{\partial^2 C}{\partial x_p \partial x_q},$$

$$K_{pq} = \mu\frac{\partial Y}{\partial x_p}\frac{\partial Y}{\partial x_q} - \frac{\partial \lambda}{\partial x_p}\frac{\partial Y}{\partial x_q} - \frac{\partial \lambda}{\partial x_q}\frac{\partial Y}{\partial x_p} - \lambda\frac{\partial^2 Y}{\partial x_p \partial x_q} + \frac{\partial^2 B}{\partial x_p \partial x_q}.$$

The $n(n+1)$ quantities H, K, thus depend on the four arbitrary functions λ, μ, B, C.

The above relations may also be written

$$H_{pq} = \frac{\partial^2 A}{\partial x_p \partial x_q} + Y\frac{\partial^2 B}{\partial x_p \partial x_q} + \lambda\frac{\partial Y}{\partial x_p}\frac{\partial Y}{\partial x_q},$$

$$K_{pq} = \frac{\partial^2}{\partial x_p \partial x_q}(B - \lambda Y) + Y\frac{\partial^2 \lambda}{\partial x_p \partial x_q} + \mu\frac{\partial Y}{\partial x_p}\frac{\partial Y}{\partial x_q}.$$

BRYN MAWR,
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