Next, the form of the invariant multiplier is investigated, and it is shown that the eliminant of $C_{m,m}$ and $C_{p,m+1}$ is a perfect mth power, of which I_{p+q-k} is the mth root. Lastly, the special cases of the quintic and septimic are discussed. The first case is treated exhaustively; for the second, on the other hand, only a summary of the main results obtained, without proofs, is given.

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F. N. COLE,
Secretary.

ON THE GROUPS GENERATED BY TWO OPERATORS SATISFYING THE CONDITION $s_1s_2=s_2^{-2}s_1^{-2}$.

BY PROFESSOR G. A. MILLER.

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§ 1. *Introduction.*

REMARKABLE general properties may easily be proved as regards the system of groups generated by two operators s_1, s_2 which satisfy the condition expressed by one of the following pair of equivalent equations:

$$s_1s_2=s_2^{-2}s_1^{-2}, \quad s_1^2s_2^2=s_2^{-1}s_1^{-1}.$$

From the facts that $(s_2s_1)^2=s_2s_1s_2s_1=s_2^{-1}s_1^{-1}=(s_1s_2)^{-1}$ and that s_1s_2 is of the same order as s_2s_1, it results that the order of s_1s_2 is an odd number. From the same equations it follows that s_1s_2 is transformed into a power of itself both by s_1 and by s_2, and hence the cyclic group generated by s_1s_2 is invariant under the group G generated by s_1, s_2. As G is generated both by s_1s_2, s_1 and also by s_1s_2, s_2 it results from the preceding sentence that if s_1^{α} is the lowest power of s_1 which occurs in the cyclic group generated by s_1s_2 it is necessary that s_2^{α} is the lowest power of s_2 occurring in this group and vice versa. Moreover, if s_1 is commutative with $(s_1s_2)^n$ the following equations are satisfied:

$$(s_2s_1)^n=(s_1s_2)^n=(s_2s_1)^{-2n} \quad \text{since} \quad (s_2s_1)^2=s_2^{-1}s_1^{-1}=(s_1s_2)^{-1}.$$

As similar remarks evidently apply to s_2, we have proved the following theorem:

If two operators satisfy the condition $s_1s_2 = s_2^{-2}s_1^{-2}$, *either they are of the same order, or the order of one is three times that of the other.*

§ 2. *Powers of* s_2s_1 *and* s_1s_2.

The following equations may readily be verified:

$$(s_2s_1)^2 = s_2s_1s_2s_1 = s_2^{-1}s_1^{-1} = s_1^2s_2^2, \qquad (s_2s_1)^3 = s_1^2s_2^3s_1,$$
$$(s_2s_1)^4 = s_1^2s_2^3s_1s_2s_1 = s_1^2s_2s_1^{-1} = s_1^2 \cdot s_2s_1 \cdot s_1^{-2}.$$

From the last of these it follows that

$$(s_2s_1)^{4n} = s_1^2(s_2s_1)^n s_1^{-2}.$$

Combining this equation with $(s_2s_1)^2 = s_1^2s_2^2$, there results the formula $(s_2s_1)^{4n+2} = s_1^2(s_2s_1)^n s_2^2$. By giving n successively the previous values of $4n+2$, we arrive at the equations

$$(s_2s_1)^2 = s_1^2s_2^2, \quad (s_2s_1)^{10} = s_1^4s_2^4, \quad (s_2s_1)^{42} = s_1^6s_2^6, \quad (s_2s_1)^{170} = s_1^8s_2^8,$$

etc. In general, it results by induction that

$$(s_2s_1)^{2/3(4^m-1)} = s_1^{2m}s_2^{2m}. \qquad (A)$$

Since $4^m \equiv 1 \bmod 3$, it follows that $\frac{2}{3}(4^m-1)$ is always an even integer, hence it must be divisible by double the order of s_2s_1 whenever the orders of s_1, s_2 divide $2m$. In particular, if the order of each of the operators s_1, s_2 is 4, G is the metacyclic group of order 20.*

In a similar way it is easy to find general formulas for powers of s_1s_2, as follows:

$$(s_1s_2)^3 = s_1s_2s_1s_2s_1s_2 = s_1s_2^{-1}s_1^{-1}s_2 = s_1^3s_2^3,$$
$$(s_1s_2)^4 = s_1^3s_2s_1^{-2} = s_1^2 \cdot s_1s_2 \cdot s_1^{-2}.$$

Hence

$$(s_1s_2)^{4n} = s_1^2(s_1s_2)^n s_1^{-2} \quad \text{and} \quad (s_1s_2)^{4n-1} = s_1^2(s_1s_2)^n s_2^2.$$

As before, we assign to n successively the values obtained from the form $4^n - 1$ and thus arrive at the equations

$$(s_1s_2)^3 = s_1^3s_2^3, \quad (s_1s_2)^{11} = s_1^5s_2^5, \quad (s_1s_2)^{43} = s_1^7s_2^7, \quad (s_1s_2)^{171} = s_1^9s_2^9,$$

etc. The general formula, which may easily be proved by in-

* Cf. Netto, *Crelle*, vol. 128 (1905), p. 254.

duction, is

$$(s_1s_2)^{\frac{2}{3}4^m+\frac{1}{3}} = s_1^{2m+1}s_2^{2m+1}. \qquad (B)$$

Since $\frac{2}{3}4^m+\frac{1}{3} = 4^m - \frac{1}{3}(4^m - 1) = 4^m -$ an odd integer, it is clear that the exponent s_1s_2 is odd for every value of m in this general formula. That is, formula A exhibits a relation between the products of even powers of s_1, s_2 and even powers of s_2s_1, while formula B is restricted to odd powers.

§ 3. *Conclusion.*

From the Introduction and formulas A, B it results that the order of G is a finite number whenever the orders of s_1, s_2 are finite; and that all the operators of G may be represented in the form $s_1^{\alpha}(s_1s_2)^{\beta}$. It is, however, not always possible to represent these operators in the form $s_1^{\alpha}s_2^{\beta}$, as is evident from the metacyclic group mentioned in § 2. Suppose that s_1, s_2 have the same order prime to 3, and let t be an operator of order 3 which is commutative with each of the operators s_1, s_2. As s_1, s_2 satisfy the condition $s_1s_2 = s_2^{-2}s_1^{-2}$, it is evident that s_1, $s_2t = t_1$ satisfy the condition $s_1t_1 = t_1^{-2}s_1^{-2}$ and that they generate a group whose order is three times that of G. This illustrates the fact that the order of one of the two operators s_1,s_2 may actually be three times that of the other, and hence the theorem at the end of § 1 relates to actual cases. As $(s_2s_1)^3 = s_2s_1s_2s_1s_2s_1 = s_2^{-1}s_1^{-1}s_2s_1$ and the cyclic group generated by s_2s_1 is invariant under G, it results that *the third power of every operator in the cyclic group generated by s_2s_1 is a commutator of G and all the commutators are such third powers.* Similar remarks clearly apply to s_1s_2, and this theorem could be deduced from the results proved in § 1.

It may be of interest to observe the analogy between the case under consideration and the relation $s_1s_2 = s_2^2s_1^2$, which was studied by Cayley * as early as 1878 and by others † more recently. Although this relation has received considerable attention yet it has given rise to only a few theorems of general interest and it presents many diffiulties which have not yet been overcome. On the other hand, the relation $s_1s_2 = s_2^{-2}s_1^{-2}$ gives rise to a number of general theorems and restricts G to a well-known category of groups. In each case the orders of s_1 and s_2 may be equal to each other. If they are unequal the order of

* *Messenger of Mathematics*, vol. 7 (1878), p. 188.

† Cf. Netto, *Crelle*, vol. 128 (1905), p. 251.

one must be an odd number while that of the other is twice this odd number when $s_1 s_2 = s_2^2 s_1^2$; but when $s_1 s_2 = s_2^{-2} s_1^{-2}$ it is only necessary that the order of one is three times that of the other.

UNIVERSITY OF ILLINOIS.

THE TEACHING OF MECHANICS.

An Elementary Treatise on Theoretical Mechanics. By J. H. JEANS. Ginn and Company, 1907. viii + 364 pp.

THERE are few topics in elementary mathematics that are more generally mishandled by the writers of our text books than Newton's three laws of motion. Perhaps it would be more accurate to say that the applications of the laws are generally misunderstood and that consequently the exponents of axioms which form the foundation of the mathematical science of mechanics rarely fail to make some fundamental error which destroys at the outset any hope of a logical development of the subject. The worst cases generally occur in the books which are published under the title "Physics." It frequently happens that the authors have not mastered the meaning of the laws; more frequently they show a want of care in their statements and explanations. In either case the effect on the student must be the same — a nebulous conception of the whole subject and a general impression that one can get along perfectly well in physics or engineering without bothering to understand what facts are directly observed and which can be deduced from the laws of motion. A man who wishes to rise to the higher levels of these professions must know such matters.

Perhaps it will not be altogether out of place to insist here on some points in the teaching in this country of applied mathematics or mathematical physics, whatever be the name we like to give to the science which concerns itself with the application of mathematics to problems in which space, time, and matter are supposed to be related by certain definitely stated laws. At the outset, the subject is a "pure" science in exactly the same way that pure mathematics is so, in that it rests solely on definitions and axioms which have no *necessary* relation to the phenomena of nature. Every problem attacked is an ideal, not an actual problem. The statement of the ideal problem must conform to the laws laid down if it is to fall within the