as 2π, 4π, 6π, etc., renders the number of non-equivalent classes of triangles with three fixed vertices finite, and Study's theorem is valid for all moduli.

Brief but artistic sections on analytic geometry of the plane (76 pages) and of space (72 pages) include much of value, as the elements of integration for volume, and the rotation groups of regular solids. A good index, and a full supply of clear diagrams, make this a valuable book of reference even for teachers who will read it but infrequently.

Of the third volume it is not too much to say that it contains a most valuable presentation of physical theories for mathematical teaching. That it is kept free from overloading of theory is seen perhaps in the fact that continuity and discontinuity are not mentioned in the index, nor critical states of matter. Half the volume is physics, vector geometry, analytical statics, dynamics, electricity and magnetism, and electromagnetism. Of the remainder, maxima and minima in geometry and capillarity fill 43 pages; probability and least squares, 40 pages; and a full and suggestive book on graphical statics the concluding 240 pages.

In a note appended to this volume, H. Weber reverts to the Mengenlehre of the first volume, cites Russell's paradox on the class of classes that do not contain themselves (which he identifies with one of Kant's antinomies); and gives an outline discussion of finite aggregates, free from objections, as he believes. These volumes certainly constitute a valuable work for every reference library. H. S. WHITE.

Leçons sur l'Intégration et la Recherche des Fonctions Primitives. Par HENRI LEBESGUE. Paris, Gauthier-Villars, 1904. 8vo. viii + 138 pp.

SINCE the publication of Lebesgue's thesis in 1902 the originality and power of his methods have attracted increasing attention to the field in which he and Baire have made such important contributions. They have given to the study of discontinuous functions an impulse which is apparent on the most cursory survey of current mathematical periodicals, and of such recent treatises as those of Young and Hobson.

The present volume, one of the series of monographs published under the direction of Borel, reproduces a course of twenty lectures delivered at the Collège de France on the Peccot foundation. Within such limits one could hardly expect a complete treatment of so large a subject as the title suggests, and in fact the author restricts himself to real functions of a single real variable. The only applications considered are connected with the problem of primitive functions and the rectification of curves, the author's researches on trigonometric series being left for a subsequent volume of the same collection. Even with this narrowing of the field the material seems almost too abundant and the reader arrives at the end of each chapter almost out of breath from the rapid pace which the author sets. One is inclined to wonder whether this haste of treatment did not extend to the editing of the volume. There is an unusual number of misprints and inaccuracies in formulas, particularly in the last chapter, and in three or four instances rather extended proofs are faulty. Lebesgue has himself recently indicated some important corrections in the *Atti della Reale Accademia dei Lincei* (series 5, classe di scienze fisiche, etc., volume 15, 1906). It should be added however, that the theorems concerned are true as stated; the inaccuracies are in the proofs. In some places brevity interferes with clearness, particularly in statements where conditions must be supplied by the reader. But, apart from these minor defects, one cannot but feel that this is a brilliant piece of work, full of originality, ingenious in its methods, important in its results.

To a certain extent the historical development of the subject is followed, beginning with definitions of integration deduced from the work of Cauchy and Dirichlet and terminating with an exposition of the method developed by the author. Everywhere the theory of point sets is of fundamental importance, and in particular the notion of the measure of a set of points. By a happy extension and simplification of Borel's definition Lebesgue gives to this term a meaning which is especially adapted to functional operations involving an enumerable infinity of elements. This may be briefly indicated as follows for a linear set E belonging to a segment AB: The points of E are enclosed in a set of intervals whose number is finite or enumerably infinite;* if we consider the sum, or the limit of the sum, of the lengths of these intervals it is obvious that for all possible sets enclosing E this sum has a lower limit $m_e(E)$; this is called the exterior measure of E. The interior measure

* If we omit the words "or enumerably infinite" this becomes the definition of measure in Jordan's sense. Lebesgue's measure includes both Jordan's and Borel's.

$m_i(E)$ is defined as the difference between the length AB and the exterior measure of the set of all points of AB which do not belong to E. If the exterior and interior measures are equal the set is said to be measurable, its measure being the common value of $m_e(E)$ and $m_i(E)$. The extension to sets in more than one dimension is obvious. The generality of this definition may be inferred from the fact that no non-measurable sets have yet been constructed by processes which do not involve such "idealistic" notions as are involved in an arbitrary choice among an infinite number of elements, though existence proofs of the latter kind have been given recently by Lebesgue, Vitali, and Van Vleck. The first-named has, however, constructed sets that are not measurable in Borel's sense.

A first application of the idea thus introduced is to be found in the formulation of the condition for the existence of an integral in Riemann's sense which requires, in the case of a limited* function, that its points of discontinuity in the integration interval form a set of measure zero. But the chief importance of this notion for Lebesgue's work arises from the part it plays in his definition of the integral, a definition which applies to all limited measurable functions, *i. e.*, functions such that for any two numbers α, β the points on the x-axis in the interval $[a, b]$ for which $\alpha < f(x) < \beta$ form a measurable set. The generality of this category of functions is apparent from what is said above as to the existence of non-measurable sets. With minor changes of notation Lebesgue's definition may be thus given: Divide the interval of variation of $f(x)$ into n subintervals by means of the numbers

$$y_0 < y_1 < y_2 < \cdots < y_n = Y,$$

and designate by $m(E_i)$ the measure of the set of points on the segment $[a, b]$ of the x-axis for which $y_{i-1} < f(x) < y_i$, and by $m(E'_i)$ the measure of the set for which $f(x) = y_i$. Then the common limit as n becomes infinite of the two sums

$$S = \sum_{i=1}^{i=n} y_i m(E_i) + \sum_{i=o}^{i=n} y_i m(E'_i), \quad s = \sum_{i=1}^{i=n} y_{i-1} m(E_i) + \sum_{i=0}^{i=n} y_i m(E'_i)$$

is the integral in Lebesgue's sense of $f(x)$ from a to b.

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* Lebesgue uses the term limited for functions all of whose values in the interval considered lie between two constants A and B; a function is finite if it has a finite value at every point of the interval.

If $f(x)$ is integrable in Riemann's sense then this new definition of the integral coincides with the ordinary one. Perhaps the relation of these two ideas is most clearly shown by means of the geometric conception of the integral as the difference between the two-dimensional measure of the set of points in $[a, b]$ having positive ordinates for which $0 < y < f(x)$, and that of the set of points for which these signs of inequality must be reversed. If these measures exist according to Jordan's definition, their difference is the integral in Riemann's sense; the insertion of the words "or enumerably infinite" in the right place in the definition of measure gives us the Lebesgue integral.

In so brief a notice as this it is impossible even to indicate the many applications which the author makes of this new and powerful instrument of analysis. There still remain unsolved cases of the problem of the primitive function, *i. e.*, the problem of determining a function whose derivative is given, but the solution is found in many cases where the ordinary integral cannot be used. In particular the question is settled whenever the given function is limited, or when it is known that the primitive function must be of limited variation. And in the case of rectifiable curves the Lebesgue integral gives their length whenever the functions $x(t)$, $y(t)$, $z(t)$ which define the curve have limited derivative numbers.

The treatment of several topics connected with the main subject should be mentioned, especially the chapters and sections on functions of limited variation and on derivative numbers. The volume closes with an admirably clear and concise note on point sets and transfinite numbers. D. R. CURTISS.

NOTES.

AT the meeting of the London mathematical society held on May 14 the following papers were read: By P. A. MACMAHON, "On the invariants of the general linear homogeneous transformation in two variables"; by H. HILTON, "On the order of the group of isomorphisms of an abelian group."

A NEW academy of sciences has been established in Finnland, with seat at Helsingfors. Two sections have been organized, one consisting of mathematics and the physical sciences, the other consisting of philology and philosophy.