shall be irreducible in a domain R are that $(c_1^2 - 4c_2 + 8)^{1/2}$ *be irrational, and that either* $l = [(1 + \frac{1}{2}c_2)^2 - c_1^2]^{1/2}$ *be irrational or else l rational and* $[\frac{1}{2}c_2^2 - c_1^2 - 2 \pm (c_2 - 2)l]^{1/2}$ *both irrational.*

11. The only linear fractional transformations which replace a reciprocal equation by a reciprocal equation are

$$x' = \pm \frac{\alpha x + \beta}{\beta x + \alpha} \qquad (\alpha^2 \mp \beta^2). \tag{11}$$

Then y, given by (3), undergoes the transformation

$$y' = \pm \frac{(\alpha^2 + \beta^2)y + 4\alpha\beta}{\alpha\beta y + \alpha^2 + \beta^2}. \tag{12}$$

The transformation on $\frac{1}{2}y$ is the square of (11).

UNIVERSITY OF CHICAGO,
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A NEW GRAPHICAL METHOD FOR QUATERNIONS.

BY PROFESSOR JAMES BYRNIE SHAW.

(Read before the Southwestern Section of the American Mathematical Society, November 30, 1907.)

1. ANY quaternion q may be written in the form $q = (w + xi) + (y + zi)j$. For convenience let us represent numbers of the form $w + xi$ (practically equivalent to ordinary complex numbers save in their products by j) by Greek characters, so that q may be written

$$q = \alpha + \beta j,$$

where for any number β we have $\beta j = j\bar{\beta}$, $\bar{\beta}$ being the conjugate of β.

The tensor of q is then the square root of the sum of the squares of the moduli of α, β. Also the scalar of q is $\frac{1}{2}(\alpha + \bar{\alpha})$, that is, the real part of α.

2. The product of $q = \alpha + \beta j$ and $r = \gamma + \delta j$ is

$$qr = (\alpha\gamma - \beta\bar{\delta}) + (\alpha\delta + \beta\bar{\gamma})j,$$

and also we have

$$rq = (\alpha\gamma - \bar{\beta}\delta) + (\bar{\alpha}\delta + \beta\gamma)j.$$

The sum of q and r is $q + r = \alpha + \gamma + (\beta + \delta)j$.

3. The graphical representation is as follows: Assume two independent planes, the A-plane and the B-plane, on which we represent α and β like ordinary complex numbers. These representations are the A and B projections of q, respectively. The sum of q and r has for its projections the sums of the respective projections of q and r. To represent the projections of the product qr, we find by the usual methods the products $\alpha\gamma$, $\beta\bar{\delta}$, $\alpha\delta$, $\beta\bar{\gamma}$. To find $\alpha\gamma$, γ is turned through the angle of α, then extended in the ratio $T\alpha:1$. So for the others. We then add vectorially $\alpha\gamma$ and $-\beta\bar{\delta}$; $\alpha\delta$ and $\beta\bar{\gamma}$.

4. If we consider r as fixed, and let the multiplier q be such that the extremities of its projections describe straight lines, then the extremities of the projections of qr will describe straight lines. For if

$$\alpha = \lambda_1 + t(\lambda_2 - \lambda_1), \quad \beta = \mu_1 + t(\mu_2 - \mu_1),$$

then

$$qr = [\lambda_1\gamma + t(\lambda_2 - \lambda_1)\gamma - \mu_1\bar{\delta} - t(\mu_2 - \mu_1)\bar{\delta}] + [\lambda_1\delta + t(\lambda_2 - \lambda_1)\delta + \mu_1\bar{\gamma} + t(\mu_2 - \mu_1)\bar{\gamma}]j.$$

The extremity of the A projection of qr thus describes the line

$$\rho = \lambda_1\gamma - \mu_1\bar{\delta} + t[(\lambda_2 - \lambda_1)\gamma - (\mu_2 - \mu_1)\bar{\delta}],$$

which passes through the points $\lambda_1\gamma - \mu_1\bar{\delta}$ and $\lambda_2\gamma - \mu_2\bar{\delta}$. The extremity of the B projection of qr describes the line

$$\rho = \lambda_1\delta + \mu_1\bar{\gamma} + t[(\lambda_2 - \lambda_1)\delta + (\mu_2 - \mu_1)\bar{\gamma}],$$

which passes through the points $\lambda_1\delta + \mu_1\bar{\gamma}$ and $\lambda_2\delta + \mu_2\gamma$. Further if $\alpha = e^{iwt}\alpha_1$, $\beta = e^{iwt}\beta_1$, then

$$qr = e^{iwt}(\alpha_1\gamma - \beta_1\bar{\delta}) + e^{iwt}(\alpha_1\delta + \beta_1\gamma)j.$$

Thus the projections of the product will describe circles with the same period as those of q.

5. We have $S.q\bar{r} = 0$ when $S(\alpha\bar{\gamma} + \beta\delta) = 0$, that is

$$\alpha\bar{\gamma} + \beta\delta = ti.$$

Solving for β we have for any value of α,

$$\beta = -\frac{\alpha\bar{\gamma}}{\delta} + t\frac{i}{\delta}.$$

That is, for any value of α the values of β terminate on a straight line perpendicular to $\bar{\delta}$. Likewise for any value of β the values of α terminate on the line

$$\alpha = -\frac{\beta\delta}{\bar{\gamma}} + t\frac{i}{\bar{\gamma}},$$

which is perpendicular to γ. This is equivalent to saying that if β terminates on the line through ξ at right angles to $\bar{\delta}$, α will terminate on the line through $-\xi\delta/\bar{\gamma}$ perpendicular to γ. Or in brief, to the line through θ/δ perpendicular to $\bar{\delta}$ corresponds the line through $-\theta/\bar{\gamma}$ perpendicular to γ, in the sense that any quaternion q whose projections terminate on these two lines in the A and B planes respectively, is *perpendicular* to the quaternion $r = \gamma + \delta j$.

6. The applications of this method to four-dimensional space are obvious, the A and B planes being two planes having only one point in common, the origin. The representation is in fact an adaptation of the descriptive geometry of four-dimensional space to the representation of quaternions as four-dimensional vectors, although this interpretation is not essential to the representation.

LOGIC AND THE CONTINUUM.

BY PROFESSOR EDWIN BIDWELL WILSON.

THE problem whether every set and in particular the continuum can be well ordered has attracted considerable attention since the days when G. Cantor first stated it in 1883.* In 1904 Zermelo offered an affirmative solution of the problem,† but his solution has not been generally regarded with favor and the discussion of the whole problem has now turned largely to a discussion of his solution. In a recent article he has summarized and discussed this discussion so fully that no repetition is called for at this time.‡ In entering so vast an arena of conflict, I would make no pretense of settling the dif-

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* *Mathematische Annalen*, vol. 21, p. 550.

† Zermelo, "Beweis, dass jede Menge wohlgeordnet werden kann," *Mathematische Annalen*, vol. 59, pp. 514–516.

‡ Zermelo, "Neuer Beweis für die Möglichkeit einer Wohlordnung," *ibid.*, vol. 65 (1907), pp. 107–128.