CONCERNING THE DEGREE OF AN IRREDUCIBLE LINEAR HOMOGENEOUS GROUP.

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IN the *Transactions*, volume 8 (1907), pages 107–112, the writer has considered the connection between the degree of an irreducible linear homogeneous group and its abstract group properties. The discussion was limited for the most part to groups whose orders are powers of a prime. In the present paper some of the results of the former paper are extended to other groups.

THEOREM I. *A linear homogeneous group G all of whose invariant operations are similarity substitutions and such that every invariant subgroup contains invariant operations besides identity either is irreducible or is simply isomorphic with each of its irreducible components.*

If G is reducible, suppose that it has been put into its completely reduced form. The invariant operations will not be affected by this change. If any irreducible component were not simply isomorphic with G, the subgroup of G that would correspond to identity in this component would contain invariant operations besides identity. But this is impossible since every invariant operation of G is a similarity substitution. Hence every irreducible component is simply isomorphic with G.

Let G be any group of finite order g that has a cyclic central generated by the operation h of order a (>1) and when G is written as a regular permutation group, of form

$$(x_{1,1}x_{1,2}\cdots x_{1,a})(x_{2,1}x_{2,2}\cdots x_{2,a})\cdots(x_{g/a,1}x_{g/a,2}\cdots x_{g/a,a}).$$

The linear substitution S

$$y_{i,j}=\sum_{k=1}^{a}\omega^{(k-1)(j-1)}x_{i,k}\quad(i=1,2,\cdots,g/a;\ j=1,2,\cdots,a),$$

where ω is a primitive ath root of unity, transforms G into a semi-canonical form and transforms h into its normal form.*

* *Transactions Amer. Math. Society*, loc. cit., p. 108.

In this transformed form of G the variables

$$x_{1,j},\ x_{2,j},\ \cdots,\ x_{g/a,j} \qquad (j = 1, 2, \cdots, a)$$

are transformed into linear combinations of themselves. Let G_j be the group formed by the substitutions of G as far as they affect these g/a variables. If $j-1$ is relatively prime to a, G_j is simply isomorphic with G.

If we suppose that every invariant subgroup of G contains at least one invariant operation besides identity, it follows that G_j is simply isomorphic with each of its irreducible components. We have therefore

THEOREM II. *A necessary and sufficient condition that a group all of whose invariant subgroups contain invariant operations besides identity be simply isomorphic with an irreducible group is that its central be cyclic.*

If $j'-1$ $(j' \neq j)$ is also relatively prime to a, then since the coefficients of G_j are either zero or powers of ω^{j-1} and $G_{j'}$ differs from G_j only in having $\omega^{j'-1}$ for ω^{j-1}, it follows that for every irreducible component of G_j there is a non-equivalent irreducible component of $G_{j'}$, and conversely. Moreover if $j-1$ is not relatively prime to a, G_j is not simply isomorphic with G. Hence the number of irreducible representations of G of any degree that are simply isomorphic with it is a multiple of $\phi(a)$.

If G is simply isomorphic with an irreducible group of the maximum degree * $n = \sqrt{g/a}$, then every invariant subgroup of G contains invariant operations besides identity. This follows from the fact that when the regular form of G is completely reduced every irreducible component occurs a number of times equal to its degree.†. Moreover the number of distinct representations of G that are simply isomorphic with it is $\phi(a)$.

Consider the formula ‡

$$\sum_R \chi(S^{-1}R^{-1}SR) = \frac{g}{f}\chi(S)\chi(S^{-1}).$$

Now suppose that G is a group in which every non-invariant commutator gives an invariant commutator besides identity,§

* *Transactions*, vol. 7 (1906), p. 67.

† Burnside, *Acta Mathematica*, vol. 28 (1904), p. 383.

‡ Frobenius, *Berliner Sitzungberichte*, 1896, II., p. 1362. We here indicate the order of the group by g instead of h.

§ This category of groups is contained in the one defined by the property that every invariant subgroup contains invariant operations besides identity.

and let S be a non-invariant operation of G that gives no invariant commutator besides identity. It follows from the formula just given that $\chi(S) \neq 0$. That is, in an irreducible group of the kind just described a necessary and sufficient condition that the trace (Spur) of any operation be different from zero is that the operation give no invariant commutator besides identity.*

Since the trace of every non-invariant commutator is zero and S gives no invariant commutators besides identity, we have $sf^2/g = \chi(S)\chi(S^{-1})$, where s indicates the number of operations with which S is commutative. Now

$$\sum_S \chi(S)\chi(S^{-1}) = g.\dagger$$

This gives $f^2 = g^2/(hg + \Sigma s_i)$, where h indicates the number of invariant operations of G, and s_i indicates the number of operations commutative with S_i (S_1, S_2, $\cdots$ being the non-invariant operations that give no invariant commutators besides identity). But each of the g/s_i operations of the conjugate set to which S_i belongs gives the same s_i. Therefore $f^2 = g/(h + j)$, where j is the number of sets of conjugates among the non-invariant operations of G that give no invariant commutator besides identity. This formula gives, for example, the degree of any irreducible group of the third class.

If G is of order p^m (p a prime), then $\chi(S)\chi(S^{-1}) = p^{s+2n-m}$, where S is a non-invariant operation that gives no invariant commutators besides identity, p^s is the number of operations commutative with S, and p^n is the degree of G. Hence, *in an irreducible group of order p^m such that every non-invariant commutator gives an invariant commutator besides identity, the square of the absolute value of the trace of any operation is a power of p (positive or negative).*‡

If G is any irreducible group of order p^m we have §

$$\sum_i x_i p^{2n_i} = p^{m-1}(p-1).$$

But $\sum_i x_i$ is the excess of the number of conjugate sets of G over the number of conjugate sets of $G/\overline{H}$. Hence

* Cf. *Transactions*, vol. 8 (1907), p. 110.

† Frobenius, loc. cit., pp. 1362, 1372.

‡ Cf. *Transactions*, loc. cit., p. 110.

§ Loc. cit., p. 109.

$$\sum_i x_i = \frac{(p^a + j)(p - 1)}{p}.$$

If now G can be simply isomorphic with irreducible groups of only one degree p^n, then

$$\frac{(p^a + j)(p - 1)}{p} p^{2n} = p^{m-1}(p - 1), \qquad p^{2n} = \frac{p^m}{p^a + j}.$$

Suppose now that G is simply isomorphic with irreducible groups of just two different degrees, p^{n_1} and p^{n_2} $(n_1 < n_2)$. Then

$$x_1 p^{2n_1} + x_2 p^{2n_2} = p^{m-1}(p - 1),$$

$$x_1 + x_2 = \frac{(p^a + j)(p - 1)}{p}.$$

Since x_1 and x_2 are integers, $p^a + j$ must be a power of p, say p^r. Obviously p^{2n_1} and p^{2n_2} cannot both be less than, or both greater than p^{m-r}. Therefore let

$$p^{2n_1} = p^{m-r-t_1} \quad \text{and} \quad p^{2n_2} = p^{m-r+t_2} \quad (t_1, t_2 > 0).$$

Then

$$x_2 = \frac{p^{r-1}(p - 1)(p^{t_1} - 1)}{p^{t_1+t_2} - 1}.$$

But this cannot be an integer. Hence, *a group of order a power of a prime cannot be simply isomorphic with irreducible groups of just two different degrees.*

It follows from this that if a group of order p^m is simply isomorphic with irreducible groups of different degrees, m must be greater than 9.*

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*See *Transactions*, loc. cit., p. 110.