SUBJECTIVE GEOMETRY.

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THE fact that many persons suppose that the non-euclidean and euclidean geometries contradict each other seems to call for further elucidation of the matter. Things that disagree do not necessarily contradict each other. One circumstance that leads to the prevalent obscurity in this matter is the use of diagrams and objective phraseology in the treatment of synthetic geometry. The reader always supposes that geometric figures are something he can see and handle. Also, in Euclid's mode of presenting the science, it is difficult to see what is fundamental and what derivative.

It is said that Legendre cast about for a long time to find the proof of the theorem "The sum of the three angles of a triangle is equivalent to two right angles" without success. He should have seen that his failure arose from the want of due determination in his definition. The theorem is true if the proper concepts are attached to the words "triangle," "angle" and "right angle."

Let us illustrate by an instance from algebra. The man who knows nothing about negative and complex quantity will say "No cubic equation can have more than three roots, but there are some which have less." The man who admits these sorts of quantity says "Every cubic equation has three roots." These statements disagree, but are not in contradiction. The case is the same as if they were written in different languages. We can translate the first statement so as to agree with the second by simply writing "positive real root" instead of "root."

The difference between the non-euclidean and the euclidean geometry is one of mere definition of terms. We shall see this more easily if we treat geometry as a subjective science. There are two methods for its exposition called synthetic and analytic. The latter makes use of the symbolic notation of algebra, and is more easily comprehended if the notion of a linear unit is admitted. At the end of the investigation, however, the latter can be brushed away as mere scaffolding, and what is left is magnitudinal geometry in place of mensuration.

We shall employ the analytic method. First it is necessary to understand what is meant by a manifold. Let x be a variable capable of taking all real values, and let these values be arranged in the order of their relative magnitudes. This is the manifold of one dimension. We may objectively illustrate it by the graph

$$\infty, \cdots -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5 \cdots, \infty,$$

exhibiting the two series of positive and negative integers separated by 0. The symbol ∞ here simply denotes unlimited extension. The values x may take are called the elements of the manifold. Again we may suppose another variable y, independent of x but able to take a series of values in the same way. Now let this manifold of one dimension exist in conjunction with each one of the elements of the manifold of one dimension for x; that is, there is constituted a sort of table of double entry. This is the manifold of two dimensions in reference to the two variables x and y. The element of the second manifold is formed by joining any value of x to any value of y. The manifold of two dimensions may be illustrated thus (the square of heavy dots denotes contemplated elements, the light dots indicated elements, and the whole is bordered with the symbol ∞):

$$\begin{array}{ccccccccc}
\infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\
\infty & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \infty \\
\infty & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \infty \\
\infty & \cdot & \cdot & \bullet & \bullet & \bullet & \cdot & \cdot & \infty \\
\infty & \cdot & \cdot & \bullet & \bullet & \bullet & \cdot & \cdot & \infty \\
\infty & \cdot & \cdot & \bullet & \bullet & \bullet & \cdot & \cdot & \infty \\
\infty & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \infty \\
\infty & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \infty \\
\infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty
\end{array}$$

Proceeding in like manner, and adopting a third variable z, we can form the manifold of three dimensions; and so on.

Let us use the letter M for manifold. Thus the three manifolds we have considered are designated by the symbols

$$M(x), \qquad M(x, y), \qquad M(x, y, z).$$

Now let us suppose that a, b, c denote three constant quantities, and let the first be added to every value of x, the second to every value of y, and the third to every value of z. Then it is evident we may write the identities

$$M(x + a) \equiv M(x), \quad M(x + a, y + b) \equiv M(x, y),$$
$$M(x + a, y + b, z + c) \equiv M(x, y, z).$$

The property of a manifold shown by these identities may be called the isotropy of transposition. It is important to bear in mind that we have established this property only by insisting that the boundary of the manifold is indefinitely removed.

We must here explain some metaphysical terms. The attributes of an entity are whatever may be rightfully considered as belonging or attached to it. We divide attributes into the two classes of properties and characteristics. Properties are attributes which cannot be ignored, they are essential to the constitution of the entity. But characteristics admit of choice. To apply this: a manifold has but one property, viz., that of the isotropy of transposition.

We can now consider the connection of the manifold of any number of dimensions with geometry of the same number of dimensions. Geometry is the science of space. What is space? It is the continuum which holds the manifold. Is anything contained in the continuum besides the manifold? If it is subjective geometry we are dealing with, the answer is nothing. Then the attributes of space are identical with those of the manifold. In our reasonings we may everywhere substitute the word manifold for the word space.

To make an application of what we have discovered: It has been questioned whether Euclid was warranted in employing his method of proof by superposition. There need not be any question about this, for the isotropy of transposition justifies it.

We proceed now to investigate possible characteristics for the manifold. Since the manifold is unique for a given number of dimensions, it follows that the number of independent characteristics, necessary for thoroughly defining the manifold, is equal to the number of its dimensions. We can build up the system of characteristics by starting from the manifold of one dimension. In a manifold of n dimensions, if we suppose one of the variables to be limited to a single value, and, as another step, the variable to be disregarded, the result is a manifold of $n-1$ dimensions. Then, in selecting the characteristics for our manifold, we can suppose the characteristics of the manifold of two dimensions are congruent with those of the manifold of one dimension. And, again, that the characteristics of the manifold of three dimensions are congruent with those of two dimensions, and so on. For the sole function of geometry is to ascertain the relativities of special elements of the manifold. We therefore start with the manifold of one dimension.

Let the special values of the variable x be denoted by x' and x''. Here the only possible relativities are two, viz., $x''-x'$ and $x'-x''$. But we insist that these relativities shall be divided so that one is always positive and the other shall express the idea of advance or recession. Then, calling the first distance, and denoting it by D, we might write the equation $D=\pm(x''-x')$. But, not liking the ambiguous sign, we prefer to write

$$D=\sqrt{(x''-x')^2},$$

where the radical must be taken positively. Then, similarly, for the manifold of one dimension, whose variable is y, we have

$$D=\sqrt{(y''-y')^2},$$

and, again, for one whose variable is z,

$$D=\sqrt{(z''-z')^2}.$$

How can D, for two dimensions, be made congruent with the D of one dimension? And, again, the D of three dimensions congruent with the D of two dimensions? It is plain that this will be accomplished in the simplest manner, if, in the first case, we put

$$D=\sqrt{(x''-x')^2+(y''-y')^2},$$

and, in the second case,

$$D=\sqrt{(x''-x')^2+(y''-y')^2+(z''-z')^2}.$$

For higher dimensions we shall have similar expressions. It must be borne in mind that these things are matters for choice.

What we have done suffices for the manifold of one dimension. But, for the manifold of two dimensions, we must have an additional characteristic. We call this λ, and choose to put

$$\frac{x'' - x'}{D} = F_1(\lambda), \quad \frac{y'' - y'}{D} = F_2(\lambda),$$

where F_1 and F_2 denote two functions not yet determined. We derive from these equations

$$[F_1(\lambda)]^2 + [F_2(\lambda)]^2 = 1,$$

and

$$F_1(\lambda) = \pm \sqrt{1 - [F_2(\lambda)]^2}.$$

Thus it is plain that the corresponding values of $F_1(\lambda)$ and $F_2(\lambda)$ can be tabulated in a table to single entry, that F_1 and F_2 are both periodic functions of λ to the same, but arbitrary, period, and that the phase of F_2 is a quarter of a period behind the phase of F_1. Consequently, we may put

$$F_1(\lambda) = \cos(m\lambda + \alpha), \quad F_2(\lambda) = \sin(m\lambda + \alpha),$$

where m and α are arbitrary constants. The latter are at our disposal, only m must not, for efficiency's sake, receive either of the values 0, ∞. Thus, for the sake of simplicity, we put $m = 1$, $\alpha = 0$. This comes to the same thing as saying that, what we first called $m\lambda + \alpha$, we now designate as λ. It must be remembered that we have put no restrictions on λ, it simply has the capacity of variation. Now we have

$$F_1(\lambda) = \cos\lambda, \quad F_2(\lambda) = \sin\lambda,$$

and the period of the two functions is 2π.

In the case of three dimensions, adopting a third variable θ, we choose to put

$$\frac{x'' - x'}{D} = F_1(\lambda, \theta), \quad \frac{y'' - y'}{D} = F_2(\lambda, \theta), \quad \frac{z'' - z'}{D} = F_3(\lambda, \theta),$$

whence is derived

$$[F_1(\lambda, \theta)]^2 + [F_2(\lambda, \theta)]^2 + [F_3(\lambda, \theta)]^2 = 1.$$

In attempting the solution of this functional equation, we note that it is not necessary for our purposes that we should arrive at a general solution, but only at any particular solution that is efficient. Thus we readily see that

$$F_1(\lambda, \theta) = \cos(m'\theta + \beta)\cos(m\lambda + \alpha),$$
$$F_2(\lambda, \theta) = \cos(m'\theta + \beta)\sin(m\lambda + \alpha),$$
$$F_3(\lambda, \theta) = \sin(m'\theta + \beta),$$

where m, m', α, β are arbitrary constants, which may have values adopted at will, except that m and m' must not receive either of the values 0, ∞. For like reasons as before, we assume $m = m' = 1$, $\alpha = \beta = 0$. Thus we have

$$\frac{x'' - x'}{D} = \cos\theta\cos\lambda, \quad \frac{y'' - y'}{D} = \cos\theta\sin\lambda, \quad \frac{z'' - z'}{D} = \sin\theta.$$

These formulas are purposely made to be congruent with those for two dimensions; if we make $\theta = 0$, and then disregard it and z, the formulas reduce to those for two dimensions.

It is not deemed necessary to notice the formulas for higher dimensions than three; they can be found in Jacobi, Vorlesungen über Dynamik, page 185.

We must notice the limits between which the new variables D, λ, θ play. While the old variables x, y, z play each between $-\infty$ and $+\infty$, D plays between 0 and $+\infty$, λ between 0 and 2π and θ between $-\pi/2$ and $+\pi/2$. It is not forbidden to extend λ and θ beyond these limits, but it must be borne in mind that the consequence of doing this is to repeat the field.

After this exposition it is easy to point out what distinguishes non-euclidean from euclidean geometry. In the first place the non-euclidean geometer insists that there may be something else in the continuum, which space is, beside the manifold; but does not define what it is. Of course it is = objective geometry — subjective geometry. In the second place he conceives that this vague something takes away from him the liberty of putting $m = 1$ in the argument $m\lambda + \alpha$. Accordingly he carries through all his ratiocinations in geometry allowing m to stand indeterminate. This course introduces no error into his science any more than the introduction of $i = \sqrt{-1}$ into arithmetic and algebra made those sciences erroneous. But it ought to be recognized as an innovation, to be justified by the advantages gained. With regard to $\sqrt{-1}$ it is almost the universal opinion of mathematicians that it is an improvement. But the approval of non-euclidean geometry is not so general; this may be due to its not having been so long before the public. The gist of the argument in favor of this view of the science is that it is not necessary to fix the period of the variable expressing orientation, that is the value of m. But both classes of geometers put $\alpha = 0$; why should it be prohibited to put $m = 1$? *

As further illustrating the subject, let us consider the theorem Legendre wished to establish. We must begin by a definition. A triangle is the bundle of relativities of three elements of the manifold of two dimensions. Let these elements be denoted as

$$x', y' \quad \text{and} \quad x'', y'' \quad \text{and} \quad x''', y'''.$$

Then we have the equations

$$(x'' - x') + (x''' - x'') + (x' - x''') = 0,$$
$$(y'' - y') + (y''' - y'') + (y' - y''') = 0.$$

We may adopt D, λ and D', λ' and D'', λ'', such that

$$x'' - x' = D\cos\lambda, \quad x''' - x'' = D'\cos\lambda', \quad x' - x''' = D''\cos\lambda'',$$
$$y'' - y' = D\sin\lambda, \quad y''' - y'' = D'\sin\lambda', \quad y' - y''' = D''\sin\lambda'',$$

where the λ are taken between the limits 0 and 2π. Consequently

$$D\cos\lambda + D'\cos\lambda' + D''\cos\lambda'' = 0,$$
$$D\sin\lambda + D'\sin\lambda' + D''\sin\lambda'' = 0.$$

From these equations may be derived the value of each of the distances expressed in terms of the others and the λ's. The equations determining these values can be written

$$D^2 = D''^2 - 2D''D'\cos[\pm\pi\pm(\lambda''-\lambda')] + D'^2,$$
$$D'^2 = D^2 - 2DD''\cos[\pm\pi\pm(\lambda-\lambda'')] + D''^2,$$
$$D''^2 = D'^2 - 2D'D\cos[\pm\pi\pm(\lambda'-\lambda)] + D^2.$$

The ambiguous signs in the arguments of the cosines are to be so taken that each argument may result between the limits 0 and π. As there are six ambiguous signs, there are no less than 64 different combinations to be selected from. By a few simple considerations we get rid of the superfluous ones. We

* It is a curious psychological commentary on this matter that the astronomer, who has a personal equation in measuring angles of position, does unconsciously what both classes of geometers consciously refrain from doing.

can denote the arguments by A, B, C, A belonging to D, B to D' and C to D''. Thence the simpler forms

$$D^2 = D''^2 - 2D''D' \cos A + D'^2,$$

$$D'^2 = D^2 - 2DD'' \cos B + D''^2,$$

$$D''^2 = D'^2 - 2D'D \cos C + D^2.$$

A, B, and C are the angles of the triangle. From the circumstance that in writing the three arguments we have used a cyclical permutation of the symbols λ, λ', λ'', it follows that we are not permitted to choose at random either of the components of the second ambiguous sign, but if we take the upper in any one case we must so take it in all, and similarly with respect to the lower. This is no more than saying that $\lambda'' - \lambda'$ belongs exclusively to D, $\lambda - \lambda''$ to D' and $\lambda' - \lambda$ to D''. Thus we may write the two cases for the expressions of the arguments

$$\pm \pi + (\lambda'' - \lambda'),\ \pm \pi + (\lambda - \lambda''),\ \pm \pi + (\lambda' - \lambda),$$

and

$$\pm \pi - (\lambda'' - \lambda'),\ \pm \pi - (\lambda - \lambda''),\ \pm \pi - (\lambda' - \lambda).$$

We may now inquire what is the sum of the three arguments or angles. Thus

$$A + B + C = \pm \pi \pm \pi \pm \pi.$$

The condition that each angle is contained between the limits 0 and π makes the sum either π or 3π. But, in order that the latter value may hold, it is necessary that $A = B = C = \pi$; this is inadmissable for the reason that some triangles exist in which one of the angles is less than π, and this condition of things must be continuous with the former state. Thus the sum of the angles would move from 3π towards π which is contrary to the equation. The value 3π must be rejected, and we have

$$A + B + C = \pi,$$

which is what Legendre was desirous of proving.

Euclid's eleventh axiom is a clumsy way of rectifying the inadequate definition of what he meant by an angle; it is clearly introduced as an afterthought. This led Legendre and many others to think that this axiom was superfluous and to attempt to construct geometries in which it was not employed. If an angle is described as a something that is attached to the vertex of a triangle, although it is shown how angles may be added, multiplied or divided, this proves nothing as to the actual magnitude of angles. The angles of a spherical triangle are also attached to the vertices and can be added, multiplied or divided in a similar way, but their actual magnitudes are different from the similar angles in a space of two dimensions. Attention ought to be directed to the fact that saying the angle is included between the sides of the triangle which come together and form the vertex adds nothing to the determination of the term "angle."

As a last point in criticism, many authors on geometry introduce misconceptions and misapprehensions into the matter by employing the concepts of the next higher dimension in a discussion which ought to be exclusively in terms belonging to the dimension dealt with. The student should be warned that lines, either straight or curved, have no existence in a space of one dimension, and that planes and curved surfaces do not exist in a space of two dimensions. This would put an end to the talk about a geometry of two dimensions on the surface of a sphere.

ON HIGHER CONGRUENCES AND MODULAR INVARIANTS.

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1. THE object of this paper is to give a two-fold generalization of Hurwitz's* explicit formula for the number of integral roots of a given congruence modulo p, p being prime. On the one hand, we may derive an equally simple formula which gives, apart from a multiple of p, the number of the roots of a specified order ($\leqq t$) of irrationality; viz., the roots belonging to the Galois field of order p^t. On the other hand, the problem may, without loss of simplicity, be further generalized †

* *Archiv der Math. u. Physik* (3), vol. 5 (1903), p. 17.

† Other generalizations of theoretical importance, but not leading readily to explicit expressions in terms of the coefficients, are given by H. Kühne, *Archiv der Math. u. Physik*, vol. 6 (1904), p. 174.