THE FOURTEENTH ANNUAL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE fourteenth annual meeting of the Society was held in New York City on Friday and Saturday, December 27–28, 1907. Two sessions were held on Friday and one on Saturday morning. On Friday evening about thirty members gathered at the informal dinner and smoker which has long been a feature of the Society's meetings. The attendance at the three sessions included the following fifty-three members:

Professor Joseph Allen, Professor William Beebe, Mr. William Betz, Professor G. A. Bliss, Professor C. L. Bouton, Professor Joseph Bowden, Professor E. W. Brown, Dr. Emily Coddington, Professor F. N. Cole, Professor T. C. Esty, Professor A. S. Gale, Dr. F. L. Griffin, Professor C. O. Gunther, Professor J. G. Hardy, Professor H. E. Hawkes, Dr. L. I. Hewes, Mr. H. R. Higley, Dr. A. A. Himowich, Dr. J. G. Hun, Professor W. H. Jackson, Mr. S. A. Joffe, Professor C. J. Keyser, Professor P. A. Lambert, Mr. W. D. Lambert, Professor W. W. Landis, Dr. N. J. Lennes, Dr. G. H. Ling, Dr. W. R. Longley, Mr. E. B. Lytle, Professor James Maclay, Professor Max Mason, Mr. C. N. Moore, Dr. R. L. Moore, Professor Frank Morley, Professor Richard Morris, Professor G. D. Olds, Professor Anna H. Palmié, Dr. H. B. Phillips, Professor James Pierpont, Miss Amy Rayson, Mr. H. W. Reddick, Professor R. G. D. Richardson, Professor P. F. Smith, Professor J. H. Tanner, Mr. C. A. Toussaint, Mr. M. O. Tripp, Professor H. W. Tyler, Professor Oswald Veblen, Mr. H. E. Webb, Professor H. S. White, Professor E. B. Wilson, Professor J. E. Wright, Professor J. W. Young.

President H. S. White presided at the morning sessions. Vice-President P. F. Smith took the chair on Friday afternoon. The Council announced the election of the following persons to membership in the Society: Mr. Charles Ammerman, McKinley High School, St. Louis, Mo.; Dr. C. S. Atchison, Williams College; Mr. B. H. Camp, Wesleyan University; Professor W. M. Carruth, Hamilton College; Mr. G. R. Clements, Williams College; Professor Julia T. Colpitts, Iowa State College; Professor J. N. Ivey, Tulane University; Professor W. H. Jackson, Haverford College; Mr. W. C. Krathwohl, Columbia University; Professor Murray Macneill, Dalhousie University; Mr. C. N. Moore, Harvard University; Professor Maria M. Roberts, Iowa State College; Mr. E. W. Sheldon, Yale University. Seven applications for membership in the Society were received.

In response to the invitation of Professor E. J. Townsend, it was decided to hold the next summer meeting of the Society at the University of Illinois. The date of the next annual meeting was fixed as Tuesday and Wednesday, December 29–30.

The reports of the Treasurer and Auditing Committee will appear in the Annual Register. The membership of the Society has increased during the past year from 554 to 580, including at present 52 life members. The number of papers presented at all meetings during the year was 166. The total attendance of members was 398. The Treasurer's report shows a balance of $6,081.82, of which $2,969.40 is credited to the life membership fund. Sales of the Society's publications during the year amounted to $1,534.

Several important changes have occurred in the past year in the administrative and editorial offices. Dr. Dennett retired in February from the Treasurer's office, and was succeeded by Professor Tanner. Professor Bôcher has succeeded Professor Moore as editor in chief of the *Transactions*. At the close of the year Professor Alexander Ziwet retired from the Committee of Publication, of which he had been a most valued member since 1893. The vacancy in the Committee was filled by the election of Professor Virgil Snyder.

At the annual election, which closed on Saturday morning, the following officers and members of the Council were chosen:

Vice-Presidents,	Professor JAMES HARKNESS,
	Professor G. A. MILLER.
Secretary,	Professor F. N. COLE.
Treasurer,	Professor J. H. TANNER.
Librarian,	Professor D. E. SMITH.

Committee of Publication,

Professor F. N. COLE,
Professor D. E. SMITH,
Professor VIRGIL SNYDER.

Members of the Council to serve until December, 1910,

Professor R. E. ALLARDICE,	Professor G. D. OLDS,
Professor MAX MASON,	Professor M. B. PORTER.

The following papers were read at this meeting:

(1) Dr. F. L. GRIFFIN: "Certain families of central orbits with a constant apsidal angle."

(2) Dr. F. L. GRIFFIN: "On the non-existence of certain types of periodic solutions in the problem of three bodies."

(3) Professor E. O. LOVETT: "On a problem in mechanics. Note II."

(4) Professor E. O. LOVETT: "On a class of periodic solutions in the problem of four bodies."

(5) Professors OSWALD VEBLEN and J. W. YOUNG: "A set of assumptions for projective geometry."

(6) Professor FRANK MORLEY: "The transformation of a Clifford configuration into itself."

(7) Professor R. G. D. RICHARDSON: "Lebesgue improper integrals."

(8) Professor E. W. BROWN: "The motion of the moon relative to a moving plane of reference."

(9) Professor E. W. BROWN: "The development of the infinite determinant."

(10) Dr. ARTHUR RANUM: "Groups of singular matrices."

(11) Professor J. G. HARDY: "Curves in a space of n dimensions."

(12) Dr. A. B. COBLE: "A reduction of the problem of solving a quintic to Klein's 'problem of the A's' by means of invariant theory."

(13) Dr. L. I. HEWES: "The necessary and sufficient condition that an ordinary differential equation shall admit a conformal group."

(14) Professor J. I. HUTCHINSON: "Hermitian forms with zero determinant."

(15) Professor E. B. WILSON: "On the uniform rotation of a homogeneous chain about a vertical axis."

(16) Professor E. B. WILSON: "On the theory of double products and strains in hyperspace.

(17) Professor MAX MASON: "Note on implicit functions."

(18) Professor H. E. HAWKES: "On the equivalence of families of bilinear forms."

(19) Professor G. A. BLISS: "An existence theorem for a partial differential equation of the first order which is non-analytic."

In the absence of the authors Professor Hutchinson's paper was presented by Professor J. W. Young, and the papers of

Professor Lovett, Dr. Ranum, and Dr. Coble were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Bertrand has discussed laws of central force for which every orbit is a closed curve.* In reality, however, he proved a far more general theorem than he stated, viz., that there are only two laws for which all the trajectories have a constant angle Θ. In Dr. Griffin's first paper, after noting this fact, consideration is given to those laws for which Θ is constant in certain families of orbits. It is shown that, while (for $n \neq 1, 2$) there is no law for which $\Theta = \pi/n$ in all the orbits, yet (for n arbitrary, rational or irrational) there exist laws of force for which $\Theta = \pi/n$ in a double infinitude of orbits, viz., in those families, the average of the nth powers of the apsidal distances of whose orbits is an arbitrary constant.

2. In a recent paper † Professor Moulton has proved the existence of a certain class of periodic plane orbits of three bodies. In this note Dr. Griffin shows that of two types closely related to these there are no orbits in three dimensions. The physical reasons for this non-existence are pointed out.

3. Professor Lovett considers the most general case of a problem studied in a paper presented to the Society at its last summer meeting (See BULLETIN, volume 14 (1907), page 57). Two differential systems F and G are to be integrated. The system F in its most general form is integrated, and the complete integration of G is achieved for those cases in which the conditions of integrability of F are realized.

As to the parameters, they must satisfy but two of the relations found in the former paper, namely $\alpha : \gamma : \delta = 9 : -45 : 40$. As regards the forces, we have the curious result that there appear no other forms than those originally found by Darboux and Halphen in the case of conic trajectories.

4. In a recent number of the *Annali di Matematica* Mr. G. Pavanini ‡ has constructed a new category of periodic solutions

* *Comptes rendus*, vol. 77, pp. 849–53.

† *Transactions*, vol. 7, pp. 537–577.

‡ Pavanini, "Sopra una nuova categoria di soluzioni periodiche nel problema dei tre corpi," *Annali di Matematica*, 3d series, vol. 13 (1906), pp. 179–203.

of the problem of three bodies. It is the object of Professor Lovett's second note to write out the corresponding solutions for those cases of the problem of four bodies which consist of three finite bodies in motion according to one of the lagrangian exact solutions of the problem of three bodies and a fourth body infinitesimally small.

5. The paper of Professors Veblen and Young contains a completely independent* set of assumptions for projective geometry, in which points and undefined classes of points called lines have been taken as the undefined elements. The assumptions are so arranged that a certain group of eight characterize what may be called general projective spaces, *i. e.*, spaces in which the points can be represented by homogeneous coordinates which are elements of a finite or infinite number system, in which the operation of multiplication may or may not be commutative. On adding to this group a ninth assumption (from which the so-called fundamental theorem of projective geometry can be derived), we obtain a set of assumptions which characterize what we call properly projective spaces, namely spaces in which the points can be represented by coordinates which are elements of a commutative number system, *i. e.*, of a finite or infinite field. Spaces in which the coordinates are elements of modular and non-modular fields are then distinguished by a further assumption. Finally it is shown how by adding other assumptions one may arrive at categorical and completely independent sets of assumptions for the projective geometries in which the coordinates are on the one hand the ordinary real numbers, and on the other hand, the ordinary complex numbers.

6. Professor Morley's paper is in abstract as follows: Clifford's theorem on p lines of a plane (Works, page 38) gives a regular configuration T_p of 2^{p-1} points and as many circles; on each point are p circles and on each circle are p points. Bifocal curves of class n, on $3n - 3$ given lines, set up a Cremona transformation between the two real foci. This gives in certain cases a transformation of the points of a Clifford configuration into themselves. Thus for T_6 the transformation is

$$xy = m + \mu t, \quad \bar{x} + \bar{y} + \bar{a} = t(x + y + a).$$

The paper will be offered to the *Transactions*.

*Ordinally independent sets have been given before; but so far as the authors are aware this is the first completely independent set.

7. In a previous paper, Professor Richardson has shown that the existence of the double and iterated integrals of any function is a sufficient condition for their equality. The present discussion covers the case where one only of the integrals is assumed to exist. By means of the notion of measure of an aggregate introduced by Borel, Lebesgue has defined * an integral which includes the ordinary (Riemann) integral as a special case. As far as is known, this integral exists for every limited function. By means of a generalization of the Lebesgue integral to the case of an unlimited function, it is shown that the existence of the double integral of any function $f(x, y)$ in a two-dimensional domain T is a sufficient condition for the equality

$$\int_T f(x, y)dT = L_Y dy \overline{\int}_{\underline{X}} f(x, y)dx,$$

where the symbol L denotes the Lebesgue integral. The existence of the integrals $\int_Y \overline{\int}_{\underline{X}} f$, $\int_Y \overline{\int}_{\underline{X}} |f|$ is a sufficient condition for the existence of the Lebesgue double integral and for the equality

$$\int_Y \overline{\int}_{\underline{X}} = L_T f.$$

Various other relations of a more general nature are derived and the results extended to the case where the domain T is unlimited.

8. Methods for partially computing the motion of the moon relative to a moving plane of reference have been given by Adams, Hill, Radau, and others. It is shown in Professor Brown's first paper that the general formulas for referring the motion to moving axes give a simple disturbing function for the general case where the periodic as well as the secular terms in the motion of the ecliptic are included.

9. In Professor Brown's second note the methods given by Hill (*Acta Mathematica*, volume 8) are replaced by the well-known formula for the development of a determinant in powers

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* Leçons sur l'intégration et la recherche des fonctions primitives, Paris, Gauthier-Villars, 1904.

of the constituents of the leading diagonal. The summing of the resulting series is made by a method which avoids the use of conditionally convergent series.

10. In order that a set of singular n-ary matrices (linear substitutions with vanishing determinants) may form a group in the ordinary sense (not in the broader sense in which the word is used by Frobenius and Schur in the *Berliner Sitzungsberichte* for 1906, I, page 209), all its matrices must have the same rank r, and their characteristic equations must have the same number of zero roots, viz., $n - r$. Every such group is reducible; in its reduced form every matrix consists of an ordinary r-ary matrix bordered with zeros. A singular matrix S will belong to some group if, and only if, the invariant factors of its characteristic equation corresponding to zero roots are linear. S belongs to a group of finite order if, and only if, the non-zero roots of its characteristic equation are roots of unity and the invariant factors of its characteristic determinant are all linear. S is the identity matrix of some group (idempotent) if, and only if, the roots of its characteristic equation are all equal to 0 or 1 and the invariant factors of its characteristic determinant are linear. Every matrix of rank n or $n - 1$ belongs to some group.

11. In a paper published in the *American Journal of Mathematics*, volume 24, Professor Hardy derived for curves in a space of four dimensions formulas corresponding to the Serret-Frenet formulas for curves in a three-dimensional space. The present paper has for its object the derivation of Serret-Frenet formulas for a curve of $n - 1$ curvatures in a space of n dimensions. The method employed is the kinematic method used by Darboux in his Théorie des surfaces, volume 1, chapter 1.

12. Except for the preliminary adjunction of the square root of the discriminant, Klein's "problem of the A's," which depends on two essential parameters, requires for its solution all the irrationalities and transcendents necessary in the solution of the general quintic. Dr. Coble shows that all the rational processes required to reduce the solution of the quintic to the problem of the A's can be set forth by invariantive methods. A form problem in which the known quantities are the values of the invariants is developed. The underlying G_{120} is a Cre-

mona group. The one necessary transformation of the quintic is accomplished by means of the linear covariants.

13. Given the differential equation $dy/dx = a(x, y)$, the problem of Dr. Hewes's paper is to determine the necessary and sufficient condition that this equation shall admit a one-parameter continuous conformal group. It is found that under the infinite group of all conformal transformations there exist, in connection with the differential equation above, invariant functions of A and its partial derivatives. There are two of the third order, from which two more of the fourth order are readily deduced. The necessary and sufficient condition is then that either of these invariants of the third order and the corresponding derived one of the fourth order are functions of the same harmonic function. If this condition is satisfied the group is uniquely determined.

14. Professor Hutchinson's paper appeared in full in the February BULLETIN.

15. The problems in the equilibrium of strings or chains which have been solved since the days when Bernoulli first investigated the catenary seem to be almost wholly those in which the field of force is rectilinear. One reason for this is probably the fact that fields other than the parallel and central fields lead to differential equations which are not linear. Professor Wilson treats the problem of equilibrium in a field compounded of gravity and a centrifugal force perpendicular to a vertical axis. This problem is readily recognized in the case of a string suspended at one end and forced to rotate at a uniform rate about a vertical axis through the point of suspension. Although the differential equation is not linear, it is of such form that it may readily be discussed qualitatively. At points considerably removed from the lowest point of the chain an approximate differential equation of the Bessel type may be obtained, and the qualitative discussion shows that in case the lower end of the string is free the configuration in the neighborhood of that point is also represented (qualitatively) to a remarkable degree by the approximate equation. One curious consequence brought out by the investigation is the existence of a rapidly increasing set of numbers $\alpha, \beta, \gamma \cdots$ such that, if the length l of the string is less than α, the string cannot depart from the vertical; whereas

if $\alpha < l < \beta$, there is one position of equilibrium and only one, if $\beta < l < \gamma$, there are two and only two positions of equilibrium, and so on. The paper will appear in the *Annals of Mathematics*, volume 9 (1908).

16. In the paper on the theory of double products and strains in hyperspace Professor Wilson presents: first, an account of so much of the late Professor Gibbs's lectures on multiple algebra as is essential to the understanding of his theory of dyadics; and second, some applications to the theory of strains and collineations in relation to involutory strains and collineations carried out by means of Gibbs's algebra. The results of this second part of the paper may be summarized briefly as follows: The determination of the square roots of the idemfactor by means of the properties of the equation of least degree; the determination of the necessary and sufficient conditions that a dyadic be resolvable into the product of two square roots of the idemfactor, and the correlation of these results with the theory of reflections in connection with unimodular strains and collineations; the introduction of certain invariant dyadics associated with any given dyadic and their application; the result that, if the Hamilton-Cayley equation of a dyadic is of least degree, the dyadic is resolvable into the product of three square roots of the idemfactor, of which the first is of the simplest type; the determination of the necessary and sufficient conditions that any dyadic be resolvable into the product of three square roots of the idemfactor.

17. The paper of Professor Mason is concerned with the determination of real implicit functions by an equation $f(x, y) = 0$ when the partial derivatives of f of order less than n vanish at the point in question. Existence theorems are obtained with the aid of Dini's theorem, after applying certain transformations similar to those used in resolving higher singularities of an algebraic curve.

18. From the properties of matrices considered as linear vector operators, Professor Hawkes proved Weierstrass's theorems, that the necessary and sufficient condition for the equivalence of two non-singular families of bilinear forms is that their elementary divisors are equivalent.

19. For a partial differential equation of the first order

$$F\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) = 0,$$

in which the first member is an analytic function of its five arguments, there exists in general through an analytic curve an analytic surface $z = z(x, y)$ which satisfies the equation. In the paper of Professor Bliss it is shown that a similar theorem is true when the function F is required to have continuous first and second derivatives only. The proof is made with the help of the theory of characteristic curves and the existence theorems for a set of ordinary differential equations

$$\frac{dx_1}{dt} = f_1(x_1, x_2, \cdots, x_n), \cdots, \frac{dx_n}{dt} = f_n(x_1, x_2, \cdots, x_n).$$

The relation between the characteristic curves and integral surfaces is well known when everything is analytic. But in order to work the other way and derive existence theorems from the properties of characteristic curves, it is necessary to use the theorems on the differentiability of solutions of a system of ordinary equations with respect to the constants of integration. These theorems seem to have been overlooked in this connection.

F. N. COLE,
Secretary.

THE DECEMBER MEETING OF THE CHICAGO SECTION.

THE twenty-second regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY was held at the University of Chicago on Monday, Tuesday and Wednesday, December 30–31, 1907, and January 1st, 1908, in connection with the fifty-eighth convocation of the American association for the advancement of science. The great gathering of scientists in other lines doubtless attracted unusual numbers of mathematicians, resulting in a wide representation of members and the largest attendance ever recorded at any meeting of the Society.

Monday afternoon and Tuesday morning were devoted to meetings with Sections A and D of the American association, for discussion of the teaching of mathematics to engineering