

$$G_{22} = \begin{vmatrix} \pm 1 & 0 & 0 & \dots \\ 0 & \pm 1 & 0 & \dots \\ \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \pm 1 \end{vmatrix}, \quad G_{23} = \begin{vmatrix} 0 & 0 & \dots & m_1 & n_1 \\ 0 & 0 & \dots & m_2 & n_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & m_r & n_r \end{vmatrix},$$

$$G_{33} = \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix},$$

in which  $m_i, n_i, \alpha, \beta, \gamma, \delta$  are integers and  $\alpha\delta - \beta\gamma = 1$ .

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## TWO TETRAEDRON THEOREMS.

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THE sphere and the tetraedron yield two combinations familiar to students of geometry, those in which one object is inscribed in the other; and one less well known, that in which the edges of the tetraedron are tangents to the sphere. A novel theorem upon the circumscribed tetraedron was propounded in 1897 by Bang and proved by Gehrke, and has been made the starting-point for extended developments by Franz Meyer\* (1903) and Neuberg (1907).† It is this: If the contact point in each face of a tetraedron circumscribed about a sphere be joined by a straight line to each vertex in its face, then three angles at each contact point are equal respectively to the three formed at any other contact point. Or it may be stated thus: Opposite edges of a circumscribed tetraedron subtend equal angles at the points of contact of the faces which contain them.

While elementary proofs of this are interesting, a more elaborate deduction is of value here as suggesting a second theorem. It can be made to depend upon the well-known theorem from the projective geometry of a straight line, namely,

\* W. Franz Meyer: "Ueber Verallgemeinerungen von Sätzen über die Kugel und das ein- resp. umbeschriebene Tetraeder." *Jahresbericht der deutschen Mathematiker-Vereinigung*, 1903, p. 137.

† J. Neuberg: "Ueber die Berührungskugeln eines Tetraeders," *Jahresbericht der deutschen Mathematiker-Vereinigung*, 1907, p. 345.

*that an involutory projective relation can be uniquely determined by two arbitrary pairs of corresponding points.*

All points, real and imaginary, of a line are represented in a one-to-one way by the real points of a plane; and thence, through stereographic projection (or inversion) by the real points of a sphere. Linear or projective transformations of points on the line are equivalent to those real linear transformations in space of three dimensions which carry the sphere over into itself, and vice versa. *These transformations do not alter the magnitude of any angle between curves on the sphere.* To apply these facts toward proof of the theorem under consideration, let the circumscribed tetraedron have vertices  $A, B, C, D$ , and for points of contact,  $A_1$  in the face  $BCD$ ,  $B_1$  in  $ACD$ , etc.

We wish to show, for example, that three angles at  $A_1$  made by rays  $A_1B, A_1C$ , and  $A_1D$  are equal respectively to three angles formed at  $B_1$  by rays drawn to  $A, D$ , and  $C$ . All these rays are of course tangent to the sphere.

Determine an involution by the pairs of points  $A_1$  and  $B_1, C_1$  and  $D_1$ . It will carry planes into planes, and planes tangent to the sphere into other such planes; in particular the four faces of the tetraedron are permuted among themselves, so that the vertices  $A$  and  $B$  are exchanged, likewise  $C$  and  $D$ . Hence the angles on the spherical surface measured by  $BA_1C, CA_1D, DA_1B$  are transformed into angles of unchanged magnitude measured respectively by  $AB_1D, DB_1C, CB_1A$ , and this proves the theorem.

This same method may be applied to draw new relations from the theorem that a collineation can be found which shall transform three given points of a straight line into three arbitrary points. Upon the sphere let the six edges of a tetraedron be tangent in six points, which may be denoted most clearly by double indices, 12, 13, 14, 34, 42, 23; *e. g.*, the edge  $AB$  shall touch the sphere at 12, etc. Through each point of contact pass a plane containing the opposite edge. We shall prove that *these planes form three pairs which intersect the sphere in orthogonal pairs of circles; and that the right angles between these circles at any contact are bisected by the traces of two other planes, each containing four points of contact.*

The relations appear on inspection as true for the regular tetraedron; and to this form every set of tangent edges of an oblique tetraedron can be brought by the admissible transformations. For the three contacts 12, 23, 31 can be brought by

such a transformation into coincidence with the contacts of three base edges of a regular tetraedron. These three points determine then the vertex 4, and with it the other three contacts; not uniquely, but as one of two definite positions: an interchange of two contacts, if necessary, will bring the vertex 4 to coincidence with that of the regular tetraedron. By this collinear transformation all the planes above mentioned come to the positions of the nine correspondingly situated planes in the regular figure, thus proving the theorem.

It is to be noticed that this tetraedron whose edges all touch a sphere is not an arbitrary figure of its class, since it depends on six constants, beside the radius of the sphere and three for the center, — ten in all, while the tetraedron with four vertices free involves twelve constants. Two sufficient conditions for inscriptibility of a sphere in this sense are these, that the sum of the lengths of two opposite edges shall be the same for each of the three opposite pairs. Less symmetrically, we note that circles inscribed in any two lateral faces must be tangent to that inscribed in the base, and from this will follow the same relation in the third lateral face.

The set of six contacts of six edges of a tetrahedron with a sphere divides naturally into threes in four ways; for example, 12, 13, 14; 23, 34, 42, three in one face and three on edges having the opposite vertex in common. *Each set of three may exchange rôles with its complementary set of three*, as is seen by applying to the figure the polar reciprocal transformation with respect to the sphere, rotating each edge through a right angle to the conjugate position. The three planes through four points of contact go into poles, whose relations to the six contacts might invite attention.

Finally it may be recalled that all the constructions suggested by these figures can be carried out in the plane, upon the six contacts of four mutually tangent circles; also that the complementary sets of three points indicate an interesting problem in the covariants of a binary cubic form.

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