$$(35)\qquad \begin{aligned} 4\rho\sigma - \rho^2 - 3\sigma^2 - 4\sigma + 2\rho - 1 = 0,\\ (\rho - 1)(\rho - 1 - 3\sigma) = 0,\end{aligned}$$

the coefficient of λ^2 being zero. For $\rho = 1$, $\sigma = 0$, and the algebra is a field. For $\rho = 1 + 3\sigma$, (35_1) is satisfied; then $\kappa = -\rho$. Substituting (5′) in (4) and reducing by (3′), we find that the coefficients of λ^2 and λ vanish, and that the constant term is

$$-\sigma^2(\sigma + 1)(4d^3 + 27g^2) = 0.$$

But the second factor is not zero in view of the irreducibility of (3′). For $\sigma = 0$, the algebra is a field. For $\sigma = -1$, $\rho = -2$, and we obtain the non-field algebra

$$(36)\qquad i^2 = j, \quad ij = ji = g - di, \quad j^2 = -d^2 - 8gi + 2dj.$$

THE UNIVERSITY OF CHICAGO,
September, 1907.

ISOTHERMAL SYSTEMS IN DYNAMICS.

BY PROFESSOR EDWARD KASNER.

(Read before the American Mathematical Society, October 26, 1907.)

CONSIDER any simply infinite system of plane curves defined by its differential equation

$$(1)\qquad y' = f(x, y).$$

The ∞^2 isogonal trajectories satisfy the equation *

$$(2)\qquad y'' = (F_x + y'F_y)(1 + y'^2),$$

where

$$F = \tan^{-1} f.$$

The theorem of Cesàro-Scheffers states that the trajectories passing through a given point have circles of curvature forming a pencil. We inquire whether any hyperosculating circles exist.

---

* Primes are employed to denote derivatives with respect to x, and literal subscripts to denote partial derivatives.

The discussion is valid for the general type

$$(3)\qquad y'' = (\psi - y'\phi)(1 + y'^2)$$

which arises in dynamics * and includes (2) as a special case. Denoting the radius of curvature by r and the arc by s, we find, for the curves (3)

$$(4)\qquad r = \frac{\sqrt{1 + y'^2}}{\psi - y'\phi},$$

$$(5)\qquad \frac{dr}{ds} = \frac{(\phi_y - \phi\psi)y'^2 + (\phi_x - \psi_y + \psi^2 - \phi^2)y' - \psi_x + \phi\psi}{(\psi - y'\phi)^2}.$$

The condition for four-point contact is the vanishing of (5). Hence for a doubly infinite system of type (3) there are at each point two curves which have contact of the third order with their circles of curvature (at that point). The directions of these curves are given by

$$(6)\quad (\phi_y - \phi\psi)y'^2 + (\phi_x - \psi_y + \psi^2 - \phi^2)y' - \psi_x + \phi\psi = 0.$$

The condition that these directions be orthogonal is

$$(7)\qquad \phi_y - \psi_x = 0.$$

Applying this to the isogonal type (2), we find

$$(8)\qquad F_{xx} + F_{yy} = 0,$$

which is recognized as the condition that (1) be isothermal. We have thus obtained a general property of isogonals and a concrete test for an isothermal system.

The isogonal trajectories of any given simply infinite system of curves are such that through each point there pass two with hyperosculating circles at that point.

These circles intersect orthogonally, for every point of the plane, when and only when the given system is isothermal.

The existence of such orthogonal circles is one of the essential properties of dynamical trajectories produced by conservative forces. When condition (7) is fulfilled, the force whose components are ϕ, ψ is conservative, and the totality of ∞^3 possible

* In the study of certain loci termed velocity curves by the author.

trajectories may be decomposed into ∞^1 "natural families," one for each value of the energy constant. Such a family is defined then by an equation of type (3) with condition (7) fulfilled.

The isogonals of a simple system of curves form a "natural family" when and only when the given system is isothermal.

The explicit relation may be obtained as follows: If the work function defining the field is denoted by W, and the constant of energy is taken to be zero, the natural family is represented by

$$(9) \qquad y'' = \tfrac{1}{2}(W_y - y' W_x)(1 + y'^2)/W.$$

If this is to be identical with the isogonal system (2), we must have

$$(10) \qquad F_x = \tfrac{1}{2} W_y/W, \qquad F_y = -\tfrac{1}{2} W_x/W.$$

These are seen to be the conditions that the functions F and $\log \sqrt{W}$ be conjugate harmonic. It follows that the base system of the isogonals is isothermal, and that the corresponding field of force is defined by a work function of the form

$$(11) \qquad W = e^{2H},$$

where H is the conjugate harmonic of F, that is, $F + iH$ is a function of $x + iy$. Our result may now be recast as follows:

A natural family of dynamical trajectories can be identified with a system of isogonal trajectories when and only when the work function is the exponential of an harmonic function.

The field of force associated with a given isothermal system is determined up to a constant factor. The lines of force are the orthogonals of the equipotential lines $W =$ const. and are therefore represented by $F =$ const.; they are seen to be the slope lines of the given system, *i. e.*, the lines joining the points where the curves (1) have parallel tangents. If only the lines of force (themselves necessarily isothermal) are given, the function F may be replaced by any function of the form $c_1 F + c_2$, and the work function is determined except for a constant factor and a constant exponent.

The most immediate application of our results is to central forces varying according to the nth power of the radius vector. This is in fact the only case in which a central force can be of type (11). Using polar coordinates we have

$$(12) \qquad W = \rho^{n+1}, \quad H = \tfrac{1}{2}(n + 1) \log \rho, \quad F = -\tfrac{1}{2}(n + 1)\theta.$$

Among the orbits produced by a central force varying as the nth power of the radius vector are included the isogonal trajectories of the curves

(13) $$y' = -\tan \tfrac{1}{2}(n+1)\theta.$$

This construction yields in the case $n = -2$ (Newtonian law) parabolas with focus at the origin; in the case $n = 1$ (elastic law) equilateral hyperbolas with center at the origin; in the case $n = -5$ the circles through the origin; and in the case $n = -3$ equiangular spirals with pole at the origin.

COLUMBIA UNIVERSITY.

ON THE EQUATIONS OF QUARTIC SURFACES IN TERMS OF QUADRATIC FORMS.

BY DR. C. H. SISAM.

(Read before the American Mathematical Society, September 5, 1907.)

THE quartic surfaces whose equations are of the form

$$\phi_2(A, B, C, D) = 0,$$

where ϕ_2, A, B, C and D are quaternary quadratic forms, were the subject of a paper by H. Durrande in the *Nouvelles Annales*.* By counting the number of constants involved, Durrande concluded that the most general quartic surface could be represented by an equation of the above form. He recognized, however, that his reasoning was not rigorous.

It will here be shown that the coefficients of the quartic surface determined by this equation are not independent, but are subject to a single condition. It will also be shown that *the equation of a general quartic surface can be written in the form*

$$\theta_2(A, B, C, D, E) = 0,$$

where θ_2 is a quinary quadratic form.

Let $\phi_2 = 0$ be reduced to the form

(1) $$A^2 + B^2 + C^2 + D^2 = 0,$$

where

$$A \equiv \Sigma a_{ij} x_i x_j \qquad (i \leqq j \leqq 4)$$

* Durrande, *Nouvelles Annales*, ser. 2, vol. 9, p. 410.