s must be commutative with at least one of the substitutions of H besides identity. If a transitive group of degree n is transformed into itself by any substitution in the same letters and if the degree of this substitution is not $n-1$, then it must be commutative with at least one of the substitutions of the transitive group besides identity.

7. The object of Professor Allardice's note on the cyclide of Dupin was to show that, by means of a transformation originally due to Laguerre (see Darboux, Théorie des surfaces, volume 1, page 253), a circle may be transformed into this cyclide; and that the principal properties of the surface may be obtained geometrically by means of the transformation.

8. The relation between the radii and distance between centers giving the condition that two circles may have a simultaneously in- and circumscribed quadrilateral was obtained by various mathematicians (Fuss, Steiner, Jacobi, Cayley) in a form limited to a special case. The complete formulas are found by Dr. McDonald, incidentally giving the interpretation of certain results of the theory of elliptic functions.

9. Professor Dickson's paper appears in full in the present number of the BULLETIN.

W. A. MANNING,
Secretary of the Section.

ON QUADRATIC FORMS IN A GENERAL FIELD.

BY PROFESSOR L. E. DICKSON.

(Read before the San Francisco Section of the American Mathematical Society, September 28, 1907.)

1. WE investigate the equivalence, under linear transformation in a general field F, of two quadratic forms *

$$q \equiv \sum_{i=1}^{n} a_i x_i^2, \quad Q \equiv \sum_{i=1}^{n} \alpha_i X_i^2 \quad (a_i \neq 0,\ \alpha_i \neq 0).$$

An obvious necessary condition is that α_1 shall be representable by q, viz., that there shall exist elements b_i in F such that

$$\alpha_1 = \sum_{i=1}^{n} a_i b_i^2.$$

* Within any field F, not having modulus 2, any quadratic form of non-vanishing determinant is equivalent to one of type q.

We assume that this condition is satisfied. Applying a suitable permutation of the x_i, we may set (§ 2)

$$W_k \equiv \sum_{i=1}^{k} a_i b_i^2 \neq 0 \qquad (k = 1, \cdots, n).$$

In view of § 3, the transformation

$$x_i = b_i y_1 + W_{i-1} y_i - b_i \sum_{j=i+1}^{n} a_j b_j y_j \quad (i = 1, \cdots, n)$$

is of non-vanishing determinant and replaces q by

$$q' \equiv \alpha_1 y_1^2 + \sum_{j=2}^{n} a_j W_j W_{j-1} y_j^2.$$

By the theorem proved in § 4 by a consideration of the automorphs of q', the forms Q and q' (with like first coefficients) are equivalent in F if, and only if,

$$(1) \qquad \sum_{i=2}^{n} \alpha_i X_i^2 = \sum_{j=2}^{n} a_j W_j W_{j-1} y_j^2$$

under a transformation in F on $n - 1$ variables. Hence *the necessary and sufficient conditions for the equivalence of the two given n-ary quadratic forms q and Q are that α_1 be representable by q and that the $(n - 1)$-ary forms* (1) *be equivalent in F*. The ultimate criteria are that $\alpha_1, \alpha_2, \cdots, \alpha_n$ be representable by forms in $n, n - 1, \cdots, 1$ variables, respectively, whose coefficients are given functions of the a_i. For example, if $n = 2$, the conditions are that α_1 be representable by q and that $a_1 a_2 \alpha_1 \alpha_2$ be a square in F. If $n = 3$, the conditions are that α_1 be representable by q, α_2 by $a_1 a_2 W_2 \xi^2 + a_3 \alpha_1 W_2 \eta^2$, and that $a_1 a_2 a_3 \alpha_1 \alpha_2 \alpha_3$ be a square in F.

2. THEOREM.* *In a non-modular*† *field F there exists a form*

* *Transactions*, vol. 7 (1906), pp. 276–8. The present proof is decidedly simpler and leads to the explicit expressions (17) for the A_j, as required for the present applications.

† The proof applies to fields having a modulus p, where $p \neq 2$ and $p \geqq n$. It may be extended to apply to any finite field. To q we apply the transformation

$$x_1 = r y_1 + s y_2, \quad x_2 = t y_1 - t^{-1} a_2^{-1} r s a_1 y_2 \qquad (t \neq 0,\ s \neq 0),$$

and obtain $m y_1^2 + m a_1 s^2 a_2^{-1} t^{-2} y_2^2 + a_3 x_3^2 + \cdots$, where $m = a_1 r^2 + a_2 t^2 \neq 0$. By choice of r and t, we may give m any assigned value in a finite field.

$\sum A_i y_i^2$ *equivalent to a given form* $q = \sum a_i x_i^2$ *and having as its first coefficient* A_1 *any preassigned mark* $\neq 0$ *which is representable by* q.

By hypothesis, there exist marks b_{i1} of F such that

$$(2) \qquad \sum_{i=1}^{n} a_i b_{i1}^2 = A_1 \neq 0.$$

Not every sum of $n-1$ of the terms t_i of (2) vanishes. For if so, we consider the sum lacking t_j and the sum lacking t_k and conclude that $t_j = t_k$ and hence that the n terms are all equal and that each is not zero; but this requires that F shall have a modulus dividing $n-1$. By applying a permutation on the x's in q, we may set $\sum_{i=1}^{n-1} a_i b_{i1}^2 \neq 0$. Not every sum of $n-2$ terms of the latter vanishes, since F does not have a modulus dividing $n-2$, etc. We may therefore set

$$(3) \qquad W_k \equiv \sum_{i=1}^{k} a_i b_{i1}^2 \neq 0 \qquad (k = 1, \cdots, n).$$

Under the transformation

$$(4) \qquad x_i = \sum_{j=1}^{n} b_{ij} y_j \qquad (i = 1, \cdots, n),$$

q becomes $\sum (A_j y_j^2 + 2B_{jk} y_j y_k)$, $(j, k = 1, \cdots, n;\ k > j)$, where

$$(5) \qquad A_j = \sum_{i=1}^{n} a_i b_{ij}^2, \quad B_{jk} = \sum_{i=1}^{n} a_i b_{ij} b_{ik}.$$

To make $B_{1k} = 0$, we take

$$(6) \qquad b_{1k} = -a_1^{-1} b_{11}^{-1} \sum_{i=2}^{n} a_i b_{i1} b_{ik} \qquad (k = 2, \cdots, n).$$

We insert these values in $\Delta = |b_{ij}|$, remove the factor $a_1^{-1} b_{11}^{-1}$ from the first row, then multiply the ith row by $a_i b_{i1}$ and add to the first row, for $i = 2, \cdots, n$. We get

$$(7) \qquad \Delta = a_1^{-1} b_{11}^{-1} A_1 \Delta_{11}, \quad \Delta_{11} \equiv |b_{is}| \quad (i, s = 2, \cdots, n).$$

We introduce the abbreviations, in addition to (3),

$$(8) \qquad P_{it} = a_i a_t b_{i1} b_{t1}, \quad R_{si} = a_i (a_i b_{i1}^2 + W_{s-1}),$$

so that, in particular, $R_{ss} = a_s W_s$. We find that

$$(9) \quad R_{ss} R_{si} - P_{is}^2 = a_s W_{s-1} R_{s+1\,i}, \quad R_{ss} P_{it} - P_{is} P_{st} = a_s W_{s-1} P_{it}.$$

Eliminating the b_{1k} from $a_1 b_{11}^2 B_{jk} (2 \leqq j < k)$ by (6), we get

$$B_{jk}^{(2)} \equiv \sum_{i=2}^{n} b_{ij} \Big(R_{2i} b_{ik} + \sum_{t \neq i}^{t=2,\ldots,n} P_{it} b_{tk} \Big). \tag{10}$$

Now $R_{22} \equiv a_2 W_2 \neq 0$. To make the coefficient of b_{2j} zero, we take

$$b_{2k} = - R_{22}^{-1} \sum_{t=3}^{n} P_{2t} b_{tk} \qquad (k = 3, \cdots, n). \tag{11}$$

Set $b_{i2} = 0$ $(i > 2)$. Then $B_{2k}^{(2)}$ evidently vanishes, while

$$\Delta_{11} = b_{22} \Delta_{22}, \quad \Delta_{22} \equiv |b_{is}| \qquad (i, s = 3, \cdots, n). \tag{12}$$

Generalizing (6) and (11), we shall take

$$b_{sk} = - R_{ss}^{-1} \sum_{t=s+1}^{n} P_{st} b_{tk} \quad (k = s + 1, \cdots, n), \tag{13}$$

$$b_{il} = 0 \qquad (i > l > 1), \tag{14}$$

and then prove by induction from s to $s + 1$ that, for

$$1 < s \leqq j < k \leqq n,$$

the product of B_{jk} by a non-vanishing factor equals

$$B_{jk}^{(s)} \equiv \sum_{i=s}^{n} b_{ij} \Big(R_{si} b_{ik} + \sum_{t \neq i}^{t=s,\ldots,n} P_{it} b_{tk} \Big). \tag{15}$$

By (10) this statement is true for $s = 2$. The coefficient of b_{sj} in (15) is zero by (13), so that we may set $i > s$. From the term given by $t = s$, we eliminate b_{tk} by means of (13). Hence

$$R_{ss} B_{jk}^{(s)} = \sum_{i=s+1}^{n} b_{ij} \Big\{ (R_{ss} R_{si} - P_{is}^2) b_{ik} + \sum_{t \neq i}^{t=s+1,\ldots,n} (R_{ss} P_{it} - P_{is} P_{st}) b_{tk} \Big\}.$$

Applying (9), we get

$$R_{ss} B_{jk}^{(s)} = a_s W_{s-1} B_{jk}^{(s+1)},$$

so that the induction is complete. Thus for given values of j and k, $1 < j < k$, we may increase s and make $s > j$; then (15) vanishes by (14). Hence every $B_{jk} = 0$ $(k > j)$. Finally, if we take each $b_{ii} \neq 0$, the determinant of (4) is not zero, in view of (14) and (7) or (12).

3. From (13), in combination with (14) and (9_2), we get

$$b_{sk} = -a_s^{-1} W_{k-1}^{-1} P_{sk} b_{kk} \qquad (s < k). \tag{16}$$

Starting with the simple formulas (14) and (16), we may readily verify that, in (5), each $B_{jk} = 0$, and

$$A_j = a_j W_j W_{j-1}^{-1} b_{jj}^2 \qquad (j = 2, \cdots, n). \tag{17}$$

THEOREM. *For given elements b_{i1} satisfying (2) and (3), and any elements $b_{jj} \neq 0$ ($j > 1$), the transformation, of non-vanishing determinant,*

$$x_i = b_{i1} y_1 + b_{ii} y_i - b_{i1} \sum_{j=i+1}^{n} a_j b_{j1} b_{jj} W_{j-1}^{-1} y_j \qquad (i = 1, \cdots, n) \tag{18}$$

replaces $\sum a_i x_i^2$ by $\sum A_j y_j^2$, the A_j being given by (2), (17).

If we employ the special values $b_{jj} = W_{j-1}$ and set $b_{i1} = b_i$, we obtain the simpler results given in § 1.

4. Within the field F, let the forms

$$Q = \sum_{i=1}^{n} \alpha_i X_i^2, \quad E = \sum_{i=1}^{n} e_i y_i^2 \quad (\alpha_i \neq 0,\ e_i \neq 0) \tag{19}$$

be equivalent under the transformation

$$S: \quad X_i = \sum_{j=1}^{n} \sigma_{ij} y_j \qquad (i = 1, \cdots, n). \tag{20}$$

In view of the formulas

$$2 e_i y_i = \frac{\partial E}{\partial y_i} = \frac{\partial Q}{\partial y_i} = \sum_{j=1}^{n} 2 \alpha_j X_j \frac{\partial X_j}{\partial y_i} = 2 \sum_{j=1}^{n} \alpha_j \sigma_{ji} X_j,$$

the inverse of S is

$$S^{-1}: \quad y_i = e_i^{-1} \sum_{j=1}^{n} \alpha_j \sigma_{ji} X_j \qquad (i = 1, \cdots, n). \tag{21}$$

Eliminating the y_i from $Q = E$, we get

$$\sum_{i=1}^{n} e_i^{-1} \sigma_{ji}^2 = \alpha_j^{-1}, \quad \sum_{i=1}^{n} e_i^{-1} \sigma_{ji} \sigma_{ki} = 0 \qquad (j \neq k). \tag{22}$$

Next, let $E(y) = mE(\xi)$ under the transformation

$$(23) \qquad C: \quad \xi_i = \sum_{j=1}^{n} \gamma_{ij} y_j \qquad (i = 1, \cdots, n).$$

Replacing α_j by me_j and σ_{ji} by γ_{ji} in (21) and (22), we get

$$(24) \qquad C^{-1}: \quad y_i = me_i^{-1} \sum_{j=1}^{n} e_j \gamma_{ji} \xi_j \qquad (i = 1, \cdots, n),$$

$$(25) \qquad \sum_{i-1}^{n} e_i^{-1} \gamma_{ji}^2 = m^{-1} e_j^{-1}, \quad \sum_{i=1}^{n} e_i^{-1} \gamma_{ji} \gamma_{ki} = 0 \qquad (j \neq k).$$

Eliminating the y_i between (20) and (24), we obtain relations of the form

$$(26) \qquad X_i = \sum_{j=1}^{n} \beta_{ij} \xi_j \qquad (i = 1, \cdots, n),$$

which imply $Q = mE(\xi)$. We desire that the first of these relations shall reduce to $X_1 = \xi_1$. By (20), (23), the conditions are

$$(27) \qquad \gamma_{1j} = \sigma_{1j} \qquad (j = 1, \cdots, n).$$

In view of (22_1) and (25_1), for $j = 1$, these require that $me_1 = \alpha_1$. Conversely, if $Q = E$ under S, and if there exists in F a matrix (γ_{ij}) with the first row identical with that of S and such that $E(y) = e_1^{-1}\alpha_1 E(\xi)$ under C, then will $Q = e_1^{-1}\alpha_1 E(\xi)$ under a transformation (26) with $X_1 = \xi_1$, and therefore

$$(28) \qquad \sum_{i=2}^{n} \alpha_i X_i^2 = e_1^{-1} \alpha_1 \sum_{i=2}^{n} e_i \xi_i^2,$$

under a transformation in F on $n - 1$ variables.*

As noted in § 1, it suffices to treat the case † $e_1 = \alpha_1$. We have therefore to deal with automorphs of $E(y)$. In view of

* For $\sigma_{1i} = 0$ $(i > 1)$, (22_2) with $j = 1$ gives $\sigma_{k1} = 0$ $(k > 1)$.

† The general case presents an essential difficulty. By (24), $|C^{-1}| = m^n |C|$, so that $m^n = \Delta^{-2}$, where $\Delta = |\gamma_{ij}|$. For n odd, m must therefore be a square in F, and hence cannot, in general, be made equal to $e_1^{-1}\alpha_1$, where α_1^{-1} is an arbitrary element of the form $\sum e_i^{-1}\sigma_{1i}^2$; see ($22_1$) for $j = 1$. The case $n = 2$ is quite simple; we may take as (γ_{ij}) the matrix

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} \\ -e_2^{-1} e_1 \sigma_{12} & \sigma_{11} \end{pmatrix}, \quad e_1^{-1}\sigma_{11}^2 + e_2^{-1}\sigma_{12}^2 = \alpha_1^{-1}.$$

§ 5, we can determine a matrix (γ_{ij}) satisfying the above conditions. We therefore have the

THEOREM. *If Q and E are equivalent in F under S and if $e_1 = \alpha_1$, then Q is equivalent to $E(\xi)$ under a transformation in F with $X_1 = \xi_1$, so that $\sum_{i=2}^{n}\alpha_i X_i^2$ and $\sum_{i=2}^{n} e_i\xi_i^2$ are equivalent in F under a transformation on $n - 1$ variables.*

Conversely, if the latter forms are equivalent in F, then evidently Q and E, with $e_1 = \alpha_1$, are equivalent in F.

5. THEOREM. *For any set of solutions σ_i in F of*

$$\sum_{i=1}^{n} e_i^{-1}\sigma_i^2 = e_1^{-1}, \tag{29}$$

the quadratic form $E = \sum e_i y_i^2$ has an automorph in F which replaces y_1 by $\sum \sigma_j y_j$.

Let Y be any skew symmetric matrix, I the identity (unit) matrix, E^{-1} the inverse of the matrix of E. Let $Z = E^{-1}Y$. Then, by Cayley's theorem, E has the automorph

$$P \equiv (I + Z)^{-1}(I - Z) \quad (\Delta \equiv |I + Z| \neq 0).$$

It will suffice to take as Y the skew matrix in which the elements outside of the first row and column are all zero, while the first row is

$$0,\ c_2,\ \cdots,\ c_n.$$

Hence

$$I \pm Z = \begin{pmatrix} 1 & \pm c_2 e_1^{-1} & \pm c_3 e_1^{-1} & \cdots & \pm c_n e_1^{-1} \\ \mp c_2 e_2^{-1} & 1 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \mp c_n e_n^{-1} & 0 & 0 & \cdots & 1 \end{pmatrix},$$

$$\Delta = 1 + \kappa, \quad \kappa \equiv e_1^{-1}\sum_{i=2}^{n} e_i^{-1}c_i^2. \tag{30}$$

The first row of $(I + Z)^{-1}$ is

$$\frac{1}{\Delta},\ \frac{-c_2}{e_1\Delta},\ \frac{-c_3}{e_1\Delta},\ \cdots,\ \frac{-c_n}{e_1\Delta}.$$

Hence the first row of the product P is

$$\frac{1-\kappa}{\Delta},\ \frac{-2c_2}{e_1\Delta},\ \frac{-2c_3}{e_1\Delta},\ \cdots,\ \frac{-2c_n}{e_1\Delta}.$$

These elements are to be made equal to

$$\sigma_1,\ \sigma_2,\ \sigma_3,\ \cdots,\ \sigma_n,$$

respectively. Hence, by (30_1), we take

$$2=\Delta(1+\sigma_1),\quad \kappa=\tfrac{1}{2}\Delta(1-\sigma_1),\quad c_i=-\tfrac{1}{2}\Delta e_1\sigma_i\quad (i=2,\cdots,n).$$

Eliminating the c_i from (30_2), and applying (29), we get

$$\kappa = \tfrac{1}{4}\Delta^2 e_1 \sum_{i=2}^{n} e_i^{-1}\sigma_i^2 = \tfrac{1}{4}\Delta^2(1-\sigma_1^2).$$

Hence the conditions may be satisfied if $1+\sigma_1 \neq 0$. But for $\sigma_1 = -1$, we may first apply the automorph $y_1' = -y_1$.

6. Although not employed in the present paper, the following generalization of the preceding theorem may be noted:

THEOREM. *Let r be any positive integer $\leqq n$. If the σ_{ji} are any solutions in F of relations (22), for $j, k = 1, \cdots, r$, and $a_j = e_j$, the form E has an automorph in F which replaces y_i by $\sum_{j=1}^{n}\sigma_{ij}y_j$, for $i = 1, \cdots, r$.*

The proof by induction, based on the result of § 5, is similar to that in the *American Journal*, volume 23 (1901), page 344, for the special case of a finite field with special values of the e_i.

7. Let F be field R of all rational numbers. There exist* rational values of $b_1, \cdots, b_4$ such that $\sum_{i=1}^{4}a_ib_i^2$ equals $+1$ or -1, according as $a_1, \cdots, a_4$ are not all negative, or all negative. Hence, by § 1, any n-ary rational quadratic form of non-vanishing determinant is reducible by a linear transformation with rational coefficients to one of the forms

$$f_{p,a,b,c} \equiv \sum_{i=1}^{p} x_i^2 - \sum_{i=p+1}^{n-3} x_i^2 + ax_{n-2}^2 + bx_{n-1}^2 + cx_n^2,$$

in which a, b, c are all negative if $p < n-3$, while $f_{p,a,b,c}$ is reducible to $f_{p,\alpha,\beta,\gamma}$ if, and only if,† the ternary form

$$t_{a,b,c} \equiv ax^2 + by^2 + cz^2$$

is reducible in R to $t_{\alpha,\beta,\gamma}$ (see end of § 1).

* A simple consequence (*Transactions*, l. c., p. 279) of a theorem due to A. Meyer; cf., Bachmann, Zahlentheorie, IV_1, p. 266.

† This part of the result was not given in my former paper.