Let

$$n - m = k_1 r, \quad m - l = k_2 r$$

$$\therefore n = (k_1 + k_2)r + l, \quad m = k_2 r + l$$

and

$$x_2 = \epsilon x_1 \quad \text{where} \quad \epsilon^r = 1.$$

We may easily show that x_1 is determined by the equation

$$(\epsilon^l - \epsilon)x_1 + \epsilon^l - 1 = 0,$$

and that x_1 and x_2 must be conjugate; and the investigation may be completed as in the last case.

It is obvious that the above method may be continued so as to include equations containing any number of terms.

It may be stated in conclusion that the problem solved in the present paper is connected with the more difficult problem of determining a quantity ρ, a function of a_0 and a_1, such that there shall always be a root either of the equation $f(x) = a$ or of the equation $f(x) = b$, with modulus less than ρ, and that this latter problem is connected with the theorem of Picard, which is discussed in Dr. Landau's paper.

STANFORD UNIVERSITY,
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ON THE DISTANCE FROM A POINT TO A SURFACE.

BY PROFESSOR PAUL SAUREL.

(Read before the American Mathematical Society, April 27, 1907.)

IT is well known that in order that the distance from a given point to a given surface be a maximum or a minimum it is necessary that this distance be measured on a normal to the surface. But, so far as I know, the various possible cases have not been enumerated. This is done in the following theorem:

If P be an elliptic point of a surface, and if C_1 be the nearer and C_2 the more remote of the principal centers of curvature, the distance from a given point N of the normal to P will be a minimum if N and P lie on the same side of C_1, a maximum if N and P lie on opposite sides of C_2, and neither a minimum nor a maximum if N coincide with C_1 or C_2, or lie between them.

If P be a hyperbolic point of a surface, and if C_1 and C_2 be the principal centers of curvature, the distance from a given point N of the normal to P will be a minimum if N lie between C_1 and C_2, and neither a minimum nor a maximum if N coincide with C_1 or C_2, or lie without the interval C_1C_2.

If P be a parabolic point of a surface, and if C_1 be the principal center of curvature, the distance from a given point N of the normal to P will be a minimum if N and P lie on the same side of C_1, and neither a minimum nor a maximum if N coincide with C_1 or if N and P lie on opposite sides of C_1.

The demonstration rests upon the following theorem in plane geometry: In order that the distance from a given point to a given curve be a maximum or a minimum it is necessary that this distance be measured on a normal to the curve. The distance is a minimum if the given point N and the foot of the normal P lie on the same side of the center of curvature C, a maximum if they lie on opposite sides, and neither a minimum nor a maximum if N coincide with C.

We shall assume that this theorem is known. It may be well, however, to point out that it becomes almost intuitive if we observe that, in the immediate neighborhood of P, the curve lies outside of any tangent circle whose center lies on the same side of C as P, that it lies inside of any tangent circle whose center lies on the opposite side of C, and that it crosses the tangent circle whose center is at C.

The theorem in solid geometry follows at once from this theorem in plane geometry if we remember that for an elliptic point the centers of curvature of the normal sections lie within the interval C_1C_2, that for a hyperbolic point they lie without this interval, and that for a parabolic point they lie on that part of the normal which does not cut the surface.

It is worth noticing that the third part of the theorem can be considered as a limiting form of either the first or the second part.

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