strations are included in the statement and demonstration of the more general and simple theorem :

If $F(x, y) = 0$ is satisfied by $x = x_0$, $y = y_0$, and if $F(x, y)$ is a continuous function of x and a continuously increasing (or decreasing) function of y near (x_0, y_0), then there exists one single-valued solution $y = \phi(x)$ near (x_0, y_0) such that $y_0 = \phi(x_0)$ and $F(x, \phi(y)) = 0$; and that solution is continuous.

A discussion of the relation of this theorem to other forms was appended.

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Secretary of the Chicago Section.

ON A LIMIT OF THE ROOTS OF AN EQUATION THAT IS INDEPENDENT OF ALL BUT TWO OF THE COEFFICIENTS.

BY PROFESSOR R. E. ALLARDICE.

(Read before the San Francisco Section of the American Mathematical Society, February 23, 1907.)

AT the end of a paper by Dr. Landau,* it is shown that every equation of the form $ax^n + x + 1 = 0$ has a root whose modulus is not greater than 2, and that every equation of the form $ax^n + bx^m + x + 1 = 0$ has a root whose modulus is not greater than 8. The object of the present paper is to show that every equation of the form

$$ax^n + bx^m + cx^l + \cdots + a_1x + a_0 = 0$$

has a root whose modulus is not greater than

$$\left|\frac{a_0}{a_1}\right| \cdot \frac{n}{n-1} \cdot \frac{m}{m-1} \cdot \frac{l}{l-1} \cdots$$

whatever be the values of the coefficients $a, b, c, \cdots$; and that, for certain values of these coefficients, this limit is attained.

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* "Ueber den Picardschen Satz," *Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich*, Jahrgang 51, 1906.

It is obvious that by reason of the substitution $a_1 x = a_0 y$, we may take $a_1 = a_0 = 1$.

The method of proof lies in showing that, by taking appropriate increments of the arbitrary coefficients, we may increase the modulus of any root of the proposed equation, unless the root in question is one of a set of equal roots, the number of which is greater by one than the number of the arbitrary coefficients. The sole difficulty lies in the consideration of roots of equal modulus, but different amplitudes.

1) Consider the equation

$$ax^n + x + 1 = 0.$$

Let $\rho\alpha$, $\rho\beta$ be two roots with common modulus ρ $(\alpha \neq \beta)$; then

$$a\rho^n\alpha^n + \rho\alpha + 1 = 0, \quad a\rho^n\beta^n + \rho\beta + 1 = 0.$$

$$\therefore \rho(\alpha^n\beta - \alpha\beta^n) + \alpha^n - \beta^n = 0$$

and

$$\rho(\alpha^{-n}\beta^{-1} - \alpha^{-1}\beta^{-n}) + \alpha^{-n} - \beta^{-n} = 0,$$

whence

$$\rho(\alpha\beta - 1)(\alpha^{n-1} - \beta^{n-1}) = 0.$$

If $\alpha^{n-1} - \beta^{n-1} = 0$, it follows also that $\alpha^n - \beta^n = 0$, which is impossible; hence $\alpha\beta - 1 = 0$, or the two roots are conjugate, and a must be real.

Now, putting $x = \rho(\cos\theta + i\sin\theta)$ in the given equation, equating to zero the real and imaginary parts, taking differentials, and eliminating $d\theta$, we may easily show that it is possible to increase the modulus of each of the roots $\rho e^{\theta i}$, $\rho e^{-\theta i}$ by giving a real increment to a. The coefficient of $d\rho$ in the relation between $d\rho$ and da cannot vanish.

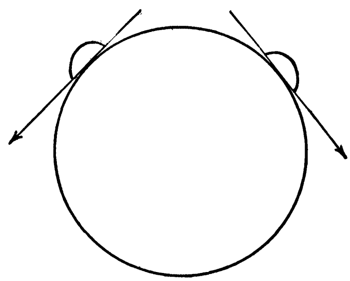
If, however, the proposed equation have two equal roots, an increment of a that will increase the modulus of one of these roots will diminish that of the other. Hence the proposed equation has always a root whose modulus is not greater than $n/(n-1)$, which is the value of the equal roots.

2) Consider now the equation

$$ax^n + bx^m + x + 1 = 0.$$

Let x_1, x_2 be two roots of equal modulus, neither of which is a multiple root; then

$$dx_1 = -\frac{x_1^n}{nax_1^{n-1}+mbx_1^{m-1}+1}da - \frac{x_1^m}{nax_1^{n-1}+mbx_1^{m-1}+1}db$$
$$= p_1 da + q_1 db, \quad \text{(say)}$$
$$dx_2 = p_2 da + q_2 db.$$



Now, representing x_1 and x_2 on a circle with radius equal to the common modulus, we see that $|x_1|$ and $|x_2|$ are increased if dx_1 and dx_2 lie in the spaces indicated in the figure.

We have

$$dx_1 = da\left(p_1 + q_1\frac{db}{da}\right), \quad dx_2 = da\left(p_2 + q_2\frac{db}{da}\right),$$

and hence, if db/da be taken arbitrarily, dx_1 and dx_2 may be made to rotate through four right angles, by varying the amplitude of da. It follows that, for some value of da, $|x_1|$ and $|x_2|$ will both be increased unless when dx_1 coincides with the tangent at x_1, dx_2 also coincides with the tangent at x_2 (in the directions indicated in the figure).

The conditions for these coincidences are

$$p_1 + q_1\frac{db}{da} = k_1 x_1 e^{-\frac{\pi}{2}i} e^{\alpha i}, \quad p_2 + q_2\frac{db}{da} = k_2 x_2 e^{\frac{\pi}{2}i} e^{\alpha i},$$

where k_1 and k_2 are real and have the same sign.

$$\therefore \frac{p_1 da + q_1 db}{p_2 da + q_2 db} = -\lambda\frac{x_1}{x_2} \quad (\lambda \text{ real and positive}).$$

Hence the fraction on the left must be independent of da and db, otherwise it would be possible to make it equal to any complex number.

$$\therefore p_1/q_1 = p_2/q_2, \quad \text{whence} \quad x_1^{n-m} = x_2^{n-m};$$
$$\therefore x_2 = \epsilon x_1, \quad \text{where} \quad \epsilon^{n-m} = 1.$$

It is easy now to deduce the values

$$x_1 = (1 - \epsilon^m)/(\epsilon^m - \epsilon), \quad x_2 = (1 - \epsilon^m)/(\epsilon^{m-1} - 1),$$

from which it follows that x_1 and x_2 are conjugate.

Denoting the conjugates of a and b by $\bar{a}$ and $\bar{b}$, we see that x_1 and x_2 are roots of the equations

$$ax^n + bx^m + x + 1 = 0, \quad \bar{a}x^n + \bar{b}x^m + x + 1 = 0$$

and hence of

$$(a\bar{b} - \bar{a}b)x^m + (a - \bar{a})x + (a - \bar{a}) = 0\,;$$

and the moduli of x_1 and x_2 may be increased together by the results obtained for the equation first considered. If a and b are both real, it may be shown as before, by putting $x = \rho(\cos\theta + i\sin\theta)$, differentiating and eliminating $d\theta$, that the moduli of x_1 and x_2 may both be increased by taking real increments of a and b.

Hence the modulus of any root of the given equation may be increased, unless it is a double root. But any double root of the given equation is a root of the equation

$$(n - m)bx^m + (n - 1)x + n = 0,$$

and may therefore have its modulus increased, unless it be a double root of this latter equation and therefore a triple root of the original equation. Thus the proposed equation always has a root whose modulus is not greater than $nm/(n-1)(m-1)$.

3) Consider now the equation

$$ax^n + bx^m + cx^l + x + 1 = 0.$$

As before, any two unequal roots of equal modulus may have their modulus increased unless

$$p_1 : q_1 : r_1 = p_2 : q_2 : r_2.$$

This leads to the equations

$$x_1^{n-m} = x_2^{n-m}, \quad x_1^{m-l} = x_2^{m-l},$$

which are impossible unless $n - m$ and $m - l$ have a common factor.

Let

$$n - m = k_1 r, \quad m - l = k_2 r$$

$$\therefore n = (k_1 + k_2)r + l, \quad m = k_2 r + l$$

and

$$x_2 = \epsilon x_1 \quad \text{where} \quad \epsilon^r = 1.$$

We may easily show that x_1 is determined by the equation

$$(\epsilon^l - \epsilon)x_1 + \epsilon^l - 1 = 0,$$

and that x_1 and x_2 must be conjugate; and the investigation may be completed as in the last case.

It is obvious that the above method may be continued so as to include equations containing any number of terms.

It may be stated in conclusion that the problem solved in the present paper is connected with the more difficult problem of determining a quantity ρ, a function of a_0 and a_1, such that there shall always be a root either of the equation $f(x) = a$ or of the equation $f(x) = b$, with modulus less than ρ, and that this latter problem is connected with the theorem of Picard, which is discussed in Dr. Landau's paper.

STANFORD UNIVERSITY,
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ON THE DISTANCE FROM A POINT TO A SURFACE.

BY PROFESSOR PAUL SAUREL.

(Read before the American Mathematical Society, April 27, 1907.)

IT is well known that in order that the distance from a given point to a given surface be a maximum or a minimum it is necessary that this distance be measured on a normal to the surface. But, so far as I know, the various possible cases have not been enumerated. This is done in the following theorem:

If P be an elliptic point of a surface, and if C_1 be the nearer and C_2 the more remote of the principal centers of curvature, the distance from a given point N of the normal to P will be a minimum if N and P lie on the same side of C_1, a maximum if N and P lie on opposite sides of C_2, and neither a minimum nor a maximum if N coincide with C_1 or C_2, or lie between them.