THE TWELFTH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE Twelfth Summer Meeting of the AMERICAN MATHEMATICAL SOCIETY was held at Williams College, Williamstown, Mass., on Thursday and Friday, September 7–8, 1905. The following twenty-eight members of the Society were in attendance:

Professor G. A. Bliss, Professor W. G. Bullard, Dr. W. H. Bussey, Professor A. S. Chessin, Professor F. N. Cole, Professor L. L. Conant, Professor D. R. Curtiss, Professor E. W. Davis, Professor W. P. Durfee, Professor F. C. Ferry, Professor W. B. Fite, Professor A. S. Gale, Professor J. G. Hardy, Professor E. V. Huntington, Professor J. I. Hutchinson, Dr. Edward Kasner, Professor Frank Morley, Professor E. D. Roe, Dr. F. H. Safford, Miss M. E. Sinclair, Dr. C. H. Sisam, Professor Virgil Snyder, Professor W. E. Story, Professor H. W. Tyler, Professor H. S. White, Professor T. W. D. Worthen, Professor J. W. A. Young, Professor J. W. Young.

Two sessions were held on Thursday, and a third session on Friday morning. Professor Frank Morley and Professor F. C. Ferry were elected chairmen. The Council announced the election of the following persons to membership in the Society: Lieutenant Colonel C. P. Echols, U. S. Military Academy; Professor G. B. Guccia, University of Palermo; Professor H. B. Evans, University of Pennsylvania; Dr. A. M. Hiltebeitel, Princeton University; Dr. J. M. Poor, Dartmouth College; Professor J. E. Williams, Virginia Polytechnic Institute. Eight applications for membership in the Society were received.

It was decided to hold the annual meeting of the Society in New York on Thursday and Friday, December 28–29. A committee, consisting of Professors E. H. Moore, James Pierpont, and G. A. Miller, was appointed to prepare and report at the October meeting a list of nominations of officers and other members of the Council to be voted for at the annual meeting.

At the close of the Thursday morning session the members were conducted through the grounds and buildings of Williams College and the collection of mathematical models was shown. On Friday afternoon the members assembled at the house of

President Hopkins and through the courtesy of the college were taken in carriages on an excursion into the Berlin mountains whose less accessible regions were traversed on foot. Several foot tours were also made on Saturday. The hospitality of the college authorities was appropriately recognized by appreciative resolutions at the close of the meeting.

The following papers were read at the meeting:

(1) Dr. W. H. BUSSEY: "Galois field tables for $p^n \leqq 169$."

(2) Dr. EDWARD KASNER: "A geometric property of the trajectories of dynamics."

(3) Professor G. A. BLISS: "A generalization of the notion of angle."

(4) Professor W. B. FITE: "Irreducible linear homogeneous groups."

(5) Dr. SAUL EPSTEEN: "Note on the structure of hypercomplex number systems."

(6) M. MAURICE FRÉCHET: "Sur l'écart de deux courbes et sur les courbes limites."

(7) Mr. RICHARD MORRIS: "On the expressibility of the automorphic functions of the group $(0, 3; l_1, l_2, l_3)$ in terms of theta series."

(8) Professor J. I. HUTCHINSON: "On certain hyperabelian functions which are expressible by theta series."

(9) Mr. N. J. LENNES: "Concerning real functions of one real variable which are completely determined over an interval by the values of the function and its derivatives for one value of the independent variable."

(10) Dr. W. A. MANNING: "On the arithmetic nature of the coefficients in groups of finite monomial linear substitutions."

(11) Dr. MAX MASON: "On the boundary value problems of linear ordinary differential equations of the second order."

(12) Professor G. A. MILLER: "On the possible number of operators of order 2 in a group of order 2^m."

(13) Professor FRANK MORLEY: "On two cubic curves in triangular relation."

(14) Dr. C. H. SISAM: "On the determination of the nodal curve on a ruled surface."

(15) Professor A. S. CHESSIN: "On the strains and stresses in a rapidly revolving circular disc."

(16) Professor L. E. DICKSON: "On the quaternary linear homogeneous groups modulo p of order a multiple of p."

(17) Professor L. E. DICKSON: "On finite algebras."

(18) Professor VIRGIL SNYDER: "On a type of rational twisted curves."

(19) Professor E. J. TOWNSEND: "Arzelà's condition for the continuity of a function defined by a series of continuous functions."

(20) Professor H. S. WHITE: "Rational plane curves as related to Riemann transformations."

(21) Professor F. R. MOULTON: "A class of periodic solutions of the problem of three bodies."

(22) Dr. C. N. HASKINS: "Note on the differential invariants of a surface and of space."

(23) Professor E. V. HUNTINGTON: "The continuum as a type of order: an exposition of the modern theory."

M. Fréchet's paper was communicated to the Society through Professor E. H. Moore, Mr. Morris's paper through Professor Hutchinson. In the absence of the authors, Mr. Morris's paper was read by Professor Hutchinson, Dr. Haskins's paper by Professor Bliss, and the papers of Dr. Epsteen, M. Fréchet, Mr. Lennes, Dr. Manning, Dr. Mason, Professor Miller, Professor Dickson, Professor Townsend, and Professor Moulton were read by title.

The papers of Dr. Bussey and Professor Townsend appeared in the October BULLETIN. Those of Dr. Kasner, Dr. Epsteen, Dr. Manning, and Professor Miller are included in the present number of the BULLETIN. Abstracts of the other papers follow below. The abstracts are numbered to correspond to the titles in the list above.

3. Geometry has to do with a set of elements called points, which is divided into subsets called lines. If it is postulated that the elements of the original set can be put into one to one correspondence with the real number ratios $x : y : z$, then the points having ratios of the form $x : y : 1$ can be represented geometrically in the ordinary cartesian plane. A curve may be defined as the set of points corresponding to two equations $x = \phi(t)$, $y = \psi(t)$, and the length of a curve as the value of a definite integral of the form

$$I = \int F\left(x, y, \frac{dx}{dt}, \frac{dy}{dt}\right) dt.$$

The nature of the geometry depends then upon the integral chosen to represent length. Hamel has used this notion in his determination of all geometries in which straight lines are the shortest distances, but without giving a corresponding generalization of angle. In Professor Bliss's paper such a generalization is developed, together with its application to surface theory and the so-called non-euclidean geometries.

4. It is well known that the number of variables in an irreducible linear homogeneous group is a divisor of the order of the group. In Professor Fite's paper this number is determined more closely for certain groups, and is definitely fixed for metabelian groups. A necessary and sufficient condition that a group whose order is a power of a prime or a group that is the direct product of such groups be simply isomorphic with an irreducible group is derived. These results for linear groups lead to certain properties of abstract groups. Finally the class of an irreducible group of order p^m in p variables (p a prime) is determined. This paper forms part of an article that is to appear in the *Transactions* under the title "Groups whose orders are powers of a prime."

6. The chief result of M. Fréchet's paper is to define a number called interspace (écart) which has the same properties in the theory of curves as distance in the theory of points: 1°. Two continuous curves C_1, C_2 being given, their interspace is a number, $(C_1, C_2) \geqq 0$, uniquely determined. 2°. The necessary and sufficient condition that two continuous curves coincide is that their interspace vanish. 3°. Given any three continuous curves C_1, C_2, C_3, then $(C_1, C_2) \leqq (C_1, C_3) + (C_3, C_2)$. Finally the concept of interspace is introduced into the theory of limiting curves.

7. Mr. Morris has studied the effect of a monodromy of the branch points of the Riemann surface $w^\nu = (z - a)^\alpha (z - b)^\beta (z - c)^\gamma (z - d)^\delta$, where $\alpha + \beta + \gamma + \delta \equiv 0 \pmod{\nu}$, on the moduli of periodicity of the integrals of the first kind. If the surface be of genus p, and if $2n$ denote the number of linearly independent integrals which are not of lower genus, it is found that the number of linearly independent parameters in terms of which the moduli of these integrals are linearly expressible is in general not less than n. Under special condi-

tions on α, β, γ, δ the minimum possible value for this number reduces to $n/2$, or $n/4$. An important consequence is that the automorphic functions of class $(0, 3; l_1, l_2, l_3)$ cannot be expressed in terms of theta series containing only one variable in any cases other than those already known.

8. In the paper by Professor Hutchinson it is shown how the moduli of the theta functions can be specialized so as to be linearly expressible in terms of a certain number of independent parameters $\zeta_1, \zeta_2, \cdots$. Certain groups of transformations on these parameters correspond to linear transformations of the theta functions and thus lead to hyperabelian functions of $\zeta_1, \zeta_2, \cdots$, which are expressible in terms of theta series.

9. In Mr. Lennes's paper the following theorem is proved: If all the derivatives of a function $f(x)$ exist for every point of a segment $a \cdots b$ then a necessary and sufficient condition that $f(x)$ shall be expressible by means of Taylor's series at every point of the segment $a \cdots b$ is that every point x_k of this segment shall lie on a segment σ_k such that for some fixed number $k_k \neq 0$

$$\mathop{\mathrm{L}}_{n=\infty} \frac{f^{(n)}(x)k_k^n}{n!} = 0$$

uniformly for x on σ_k.

This condition differs essentially from that given by Pringsheim, though the proof in some respects is similar.

It is then shown that, if for a certain point the Taylor series is divergent or is convergent but fails to represent the function, then the function is not uniquely determined on any interval by the value of the function and those of its derivatives at that point, consequently it is impossible to devise an algorithm in terms of $f^{(i)}(x_0)$ $(i = 0, \cdots, \infty)$ which shall represent the function over an interval containing x_0 in case Taylor's expansion fails.

As a lemma a necessary and sufficient condition is given that a series of continuous functions whose sum is a continuous function shall be uniformly convergent.

11. Dr. Mason considered the general boundary conditions:

$$a_1y(x_1) + a_2y(x_2) + a_3y'(x_1) + a_4y'(x_2) = A$$
$$b_1y(x_1) + b_2y(x_2) + b_3y'(x_1) + b_4y'(x_2) = B,$$

in connection with certain ordinary linear differential equations of the second order which contain a parameter. Some of the results of the paper were published in the *Comptes rendus*, April 27, 1905.

13. Professor Morley's paper is in abstract as follows : The fixed points of a collineation belonging to a pencil, in a plane, are on a cubic curve ϕ; and the fixed lines are on a cubic curve f. Thus ϕ and f admit an infinity of Poncelet triangles. By joining every point to the corresponding point in a collineation we pick out of the ∞^3 elements of the plane a nexus of ∞^2 elements. The elements common to two nexuses form the two cubics. Each triangle forms with every element a constant double ratio. The two cubics have 6 contacts, 6 simple common points, and 6 simple common lines.

14. In Dr. Sisam's paper the equation of a plane curve in one to one correspondence with the nodal curve of a rational scroll was obtained. Let the equations of the scroll be : $x_i = a_i(u) + vb_i(u)$ $(i = 1, 2, 3, 4)$. If the generators determined by $u = u$ and $u = u'$ intersect, then the determinant $|a_1(u)a_2(u')b_3(u)b_4(u')| = 0$. This equation, after division by $(u - u')^2$, may be expressed in terms of $u + u'$ and uu' only. Call it $F(u + u', uu') = 0$. Put $u + u' = \xi$, $uu' = \eta$ and consider $F(\xi, \eta) = 0$ as the equation of a plane curve. It then follows that each irreducible component of $F(\xi, \eta) = 0$ determines an irreducible component of the nodal curve on the surface. If the latter is double only, then the number of times it is cut by an arbitrary generator equals the order, and its genus equals the genus of the component of $F = 0$ which determines it.

The components of the double developable and their properties analogous to those of the double curve are also determined by the components of $F(\xi, \eta) = 0$. The equation in u found by putting $u' = u$ in $F(u + u', uu') = 0$ determines the parameters of the torsal generators.

15. If the radial and axial displacements u and w caused by the rotation of a circular disc of thickness $2h$ and radius R, at a point (v, z), be expressed as follows :

$$(1) \qquad u = -\frac{\rho\omega^2}{8(2\mu + \lambda)} r^3 + \sum_0^\infty u_n z^{2n},$$

$$w = \quad w_n z^{2n+1}; \tag{2}$$

if, moreover, we put

$$z_n = h^{2n}\left\{ (2\mu + \lambda)\frac{1}{r}\frac{dru_n}{dr} + \lambda(2n+1)w_n \right\}, \tag{3}$$

$$y^n = h^{2n}\left\{ \lambda\frac{1}{r}\frac{dru_n}{dr} + (2\mu + \lambda)(2n+1)w_n \right\}, \tag{4}$$

then we have, to determine x_n and y_n, the system of differential equations

$$\begin{aligned} &(5)\ h^2 Dx_n = (2n+1)(2n+2)y_{n+1}, \\ &(6)\ h^2 Dy_n = -(2n+1)(2n+2)(x_{n+1} + 2y_{n+1}) \end{aligned} \quad (n = 0, 1, 2, \cdots),$$

where D indicates the operation

$$\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}.$$

The surface conditions for $z = \pm h$ furnish two equations, viz.:

$$\sum \frac{x_n}{2n+1} = C, \qquad \sum y_2 = \frac{\lambda\rho\omega^2}{2(2\mu+\lambda)}r^2. \tag{7}$$

The integration of the system (5)–(6), together with equations (7), gives the following results:

$$(n > 1)\ x_n = \sum C_{n,e} J_0\left(\frac{\lambda_e}{h}r\right), \qquad (n \geqq 1)\ y_n = \sum C'_{n,e} J_0\left(\frac{\lambda_e}{h}r\right)$$

$$\begin{aligned} &x_0 = C + \sum C_{0,e} J_0\left(\frac{\lambda_e}{h}r\right), \\ &x_1 = -\frac{\lambda\rho\omega^2}{2\mu+\lambda}h^2 + \sum C_{1,e} J_0\left(\frac{\lambda_e}{h}r\right), \\ &y_0 = \frac{\lambda\rho\omega^2}{2(2\mu+\lambda)}\mu^2 + \sum C'_{0,e} J_0\left(\frac{\lambda_e}{h}r\right), \end{aligned} \tag{8}$$

where the λ_e are the roots of the transcendental equation

$$(9) \qquad \sin 2ix + 2ix = 0$$

and the coefficients $C_{n,e}$, $C'_{n,e}$ are given by the formulas

$$C_{n,e} = \left(1 + \frac{2n}{1 + \cos^2 i\lambda_e}\right)\frac{\lambda_e^{2n}}{2n\,!}\,C_{0,e},$$

$$C'_{n,e} = \left(\frac{\sin^2 i\lambda_e - 2n}{1 + \cos^2 i\lambda_e}\right)\frac{\lambda_e^{2n}}{2n\,!}\,C_{0,e},$$

while the coefficients C and the $C_{0\,e}$ are determined from the two remaining surface conditions for $r = R$.

Once the x_n and y_n are determined, the w_n are obtained from (3)–(4) directly and the u_n by a simple integration. Thus will be found the displacements u and w and by their means the strains and stresses at any point of the disc.

This solution of the problem does not readily lend itself to practical applications on account of its intricate form. A simplified solution will be the subject of a subsequent paper.

16. The group of all quaternary linear homogeneous substitutions of determinant unity modulo p is of order $p^6(p^4-1)(p^3-1)(p^2-1)$. It is shown by Professor Dickson that any subgroup of order a multiple of p^6 is contained in a group conjugate with one of three maximal subgroups

$$\begin{array}{ll}(\mathrm{I}) & (a_{ij}),\ a_{13} = a_{14} = a_{23} = a_{24} = 0,\ |a| = 1\,; \\ (\mathrm{II}) & (a_{ij}),\ a_{14} = a_{24} = a_{34} = 0,\ |a| = 1\,; \\ (\mathrm{III}) & (a_{ij}),\ a_{12} = a_{13} = a_{14} = 0,\ |a| = 1.\end{array}$$

Here II and III are simply isomorphic groups of order $p^3(p^3-1)(p^3-p)(p^3-p^2)$, while I is of order $p^6(p^2-1)^2(p-1)$. Next, let H be a subgroup of order p^5N, N prime to p, $p > 2$. If its G_{p^5} is invariant, H is contained in (a_{ij}), $a_{ij} = 0 (j > i)$ except a certain $a_{s-1,\,s}$. If G_{p^5} is not invariant, H is conjugate with one of the types: (I) with the $\binom{a_{11}\ a_{12}}{a_{21}\ a_{22}}$ defining a binary group of order prime to p; (I) with the same restriction on the $\binom{a_{33}\ a_{34}}{a_{34}\ a_{44}}$; (I) with

$$a_{11} = \pm e^{-1}a_{33},\ a_{12} = \pm \mu^{-1}e^{-1}a_{34},\ a_{21} = \pm \mu e^{-1}a_{43},\ a_{22} = \pm e^{-1}a_{44},$$

where $e = a_{33}a_{44} - a_{34}a_{43}$, and μ is a fixed integer, either 1 or a particular quadratic non-residue of p. According to the restrictions placed on the sign $\pm$ and on e, the last group is of order $p^5(p^2 - 1)t$, t any chosen factor of $2(p - 1)$. The paper has been offered to the *American Journal of Mathematics*.

17. Of the algebras in which all but one of the postulates of a field hold (*Transactions*, 1905, page 201), the most interesting are those in which the commutativity or associativity of multiplication is not assumed. In his second paper, Professor Dickson treats at length finite algebras whose elements form a group under addition, whose elements $\neq 0$ form a group under multiplication, and for which the left-hand distributive law holds. (If also the right-hand distributive law holds, the algebra is a field.) It is shown that the elements may be given the notation $(a_1, \cdots, a_n)$, each a_i an integer taken modulo p a prime, and that $(a_i) + (b_i) = (a_i + b_i)$. The algebra is a field if $p^n - 1$ and n are relatively prime, or if $n = 1$. In the remaining cases, non-field algebras exist; a typical example is

$$(a, b) \times (x, y) = (ax + evby,\ bx + eay),$$

where v is a fixed quadratic non-residue of p, and $e = \pm 1$ according as $a^2 = vb^2$ is a quadratic residue or non-residue of p.

In the second type of algebra, the elements form a group under addition, the product of any two is in the set, the right-hand identity and inverse occur, and multiplication is commutative and distributive; the omitted postulate is associativity of multiplication. If the algebra is finite the elements are $(a_1, \cdots, a_n)$, the a_i taken modulo p, prime, and $(a_i) + (b_i) = (a_i + b_i)$. If $n < 3$, the algebra is a field. Non-field algebras are found for n any multiple of 3. The remaining cases were not examined. If F is any finite or infinite field not having modulus 2, an algebra having all the required properties is that with the units $1, i, j$, and coördinates in F, where $i^2 = j$, $ij = ji = b + \beta i$, $j^2 = -\beta^2 = 8bi - 2\beta j$, where b and β are such that $x^3 = b + \beta x$ is irreducible in F. In this linear, non-associative algebra, every element has an unique inverse. When F is the field of integers modulo p, this algebra is the only such non-field algebra for $p = 3, 5, 7$ or 11, the cases examined exhaustively.

18. The curves discussed by Professor Snyder are rational, of order n, and have two tangents of $n - 1$ pointic contact at

A and B. The osculating plane at any point P cuts the curve in $n-3$ points Q_i. The surface generated by the lines PQ_i is composite, each factor being rational and of order $2n-4$. The given curve is tacnodal on each such surface, and the residual curve breaks up into $n-2$ factors, each being cut twice by every generator. If n is even, one of the points Q_k is the harmonic conjugate of P as to A and B. The lines PQ_k generate a scroll having AB for multiple directrix, upon which the given curve is an asymptotic line.

20. In the geometry of birational transformations in a plane or other rational surface, the Cremona group is a subgroup of the general algebraic or Riemann group, hence the former may have some invariants distinct from those of the latter. Professor White points out the fact that this is not so when the object to which the transformations are applied is a single rational curve on the surface.

21. Professor Moulton's paper discusses the existence, properties, and construction of certain periodic solutions of the problem of three bodies in which the orbits reduce to circles with the vanishing of certain parameters. With proper specialization of the parameters the problem is that of the periodic orbits of two mutually disturbing finite planets; with another specialization, the problem of the periodic solution of a finite satellite disturbed by the sun; and with certain further limitations, the moon's variational orbit. The results are developed literally so as to apply to any case in the various types, and numerical illustrations are exhibited. They include the moon's variational and parallactic orbit carried out to an order of accuracy which leaves nothing to be desired, and two classes of the periodic orbits discovered by Darwin (*Acta Mathematica*, volume 21) from numerical experiments.

22. Dr. Haskins's paper indicates the simplifications which the use of Ricci's methods of covariant differentiation permit to be made in Forsyth's recent work on differential invariants of quadratic differential forms.

23. The main part of Professor Huntington's paper gives a systematic elementary account of the purely ordinal theory of the continuum, as begun by Dedekind in 1872, and completed by Georg Cantor in 1895. An appendix on Cantor's "well-

ordered" classes and the transfinite numbers will serve as an introduction to the study of these most recent accessions to the list of mathematical concepts. The matter is of interest to the philosopher as well as to the mathematician; and the present exposition is intended especially for the general student of scientific method, who, without technical mathematical training, wishes to keep in touch with the modern development in the logic of mathematics. The mathematical prerequisites have been reduced to a minimum; the demonstrations are given in full; all new concepts are defined explicitly by sets of independent postulates; and in connection with each definition numerous examples are given, to illustrate, in a concrete way, not only the systems which have, but also those which have not, the property in question. The paper is being published in the *Annals of Mathematics* for July and October, 1905.*

The chapter headings are as follows: On classes in general; Ordered classes, or "series"; Discrete series, especially the type of order exhibited by the natural numbers; Dense series, especially the type of the rational numbers; Continuous series, especially the type of the real numbers; Continuous series in more than one dimension, with a note on multiply-ordered classes. An appendix treats of Cantor's "well-ordered" classes, and the transfinite numbers, and there is an index of technical terms. The paper contains also a bibliography of Cantor's writings on these subjects.

F. N. COLE,
Secretary.

A SET OF GENERATORS FOR TERNARY LINEAR GROUPS.

BY MISS IDA MAY SCHOTTENFELS.

(Read before the American Mathematical Society, September 17, 1904.)

The following is a proof that the substitutions

$$c_h: \quad x'_i = x_{i+1},\ x'_k = x_1 + hx_2 \quad (i = 1, 2, \cdots k-1;\ h = 0, 1)$$

generate (1) the ternary linear substitution group with integral

* Reprints of this and other papers published in the *Annals* can be ordered from the Publication Office of Harvard University.