The $O$-relations are wholly symmetric, and may hold for pairs, triads, or in fact for any multitude of the elements. The unsymmetric relations possible in the algebra of logic are deduced from the properties of the $O$-relations. A new theory of the nature of logical addition and multiplication is developed, founded on Kempe's, but involving a considerable generalization of his procedure. It is then shown how the axioms of geometry can be proved as theorems of the algebra of logic, when the entities to which these theorems are to be applied are chosen in a special way.

F. N. COLE,
*Secretary.*

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# THE APRIL MEETING OF THE CHICAGO SECTION.

THE seventeenth regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY was held at the University of Chicago on Saturday, April 22, 1905. The total attendance was forty-five, including the following members of the Society:

Professor G. A. Bliss, Professor Oskar Bolza, Dr. W. H. Bussey, Professor E. W. Davis, Professor L. E. Dickson, Dr. Saul Epsteen, Mr. M. E. Graber, Professor G. W. Greenwood, Professor E. R. Hedrick, Professor T. F. Holgate, Mr. A. E. Joslyn, Dr. H. G. Keppel, Professor Kurt Laves, Mr. N. J. Lennes, Dr. A. C. Lunn, Mr. J. H. Maclagan-Wedderburn, Professor Heinrich Maschke, Professor E. H. Moore, Professor F. R. Moulton, Professor H. L. Rietz, Professor J. B. Shaw, Professor H. E. Slaught, Dr. W. M. Strong, Professor H. S. White, Mr. N. R. Wilson, Dr. J. W. Young, Professor J. W. A. Young.

The first session was called to order at ten o'clock. Professor E. R. Hedrick was elected chairman, and Dr. J. W. Young secretary pro tem. On motion of Professor Moore it was voted that the chair appoint a committee to consider means of improving the meetings of the Section. The chair appointed on this committee Professors Dickson, Davis and Bliss.

The following papers were read:

(1) Dr. SAUL EPSTEEN and Mr. R. L. MOORE: "A set

of postulates for the theory of matrices (preliminary communication)."

(2) Mr. R. L. MOORE: "Sets of metrical hypotheses for geometry."

(3) Professor J. B. SHAW: "Groups of quaternions."

(4) Professor J. B. SHAW: "General algebra."

(5) Professor H. L. RIETZ: "Simply transitive primitive groups which are simple groups."

(6) Mr. N. R. WILSON: "Simple groups of even order less than 5000."

(7) Professor L. E. DICKSON: "On the cyclotomic function."

(8) Professor ALFRED LOEWY: "Ueber die vollständig reduciblen Gruppen, die in eine Gruppe linearer homogener Substitutionen gehören."

(9) Professor HEINRICH MASCHKE: "Differential parameters of the first order."

(10) Professor HEINRICH MASCHKE: "The Gaussian curvature of hyperspace."

(11) Professor G. A. BLISS: "A theorem of Volterra concerning line functions."

(12) Professor G. A. BLISS: "The inverse problem in the calculus of variations in parametric representation."

(13) Mr. A. L. UNDERHILL: "The invariantive property of the four necessary and sufficient conditions of the calculus of variations under contact transformations (preliminary communication)."

(14) Mr. M. E. GRABER: "On the definition and classification of cyclidal surfaces."

(15) Professor L. E. DICKSON: "On the equations of geometry."

(16) Dr. OSWALD VEBLEN: "Projective geometry in an arbitrary field."

(17) Dr. OSWALD VEBLEN and Dr. W. H. BUSSEY: "Finite projective geometries."

(18) Mr. J. H. MACLAGAN-WEDDERBURN: "A theorem on finite algebras."

(19) Mr. F. W. OWENS: "The Desargues theorem and the ideal elements in projective geometry (preliminary communication)."

(20) Dr. OSWALD VEBLEN and Mr. J. H. MACLAGAN-WEDDERBURN: "Non-desarguesian and non-pascalian geometries."

(21) Dr. OSWALD VEBLEN: "Note on the square root and the relations of order."

(22) Professor L. E. DICKSON: "On hypercomplex number systems."

Professor Loewy's paper was communicated to the Society through Professor Moore; Mr. R. L. Moore was introduced by Dr. Epsteen, and Mr. Owens by Dr. Veblen. In the absence of the author, Professor Loewy's paper was presented by Professor Dickson. The papers numbered 9, 16, 21, 22 were read by title. The meeting adjourned at a quarter past six o'clock.

Abstracts of the papers presented follow below. The abstracts are numbered to correspond to the titles in the list above.

1. In this paper the authors presuppose the existence of a field $F$ with its usual properties. The undefined elements are vid, matrix, $\odot$, $\otimes$, $\oplus$. The postulates are nine in number and they are independent if the field $F$ contains a mark which is not an integer. Otherwise one of the postulates is dependent upon the other eight which are themselves independent.

It is worthy of note that the associative and distributive laws of multiplication are not introduced by postulates. The former is a theorem, the latter is an easy consequence of the postulates and one of the definitions.

2. Mr. Moore's paper contains a set of independent postulates for euclidean geometry with point, order, and congruence (of segments) as undefined symbols. Congruence for angles is introduced by definition. If in place of a certain continuity postulate is substituted the postulate that every segment has a midpoint, there follows that portion of Hilbertian plane geometry which is not partly or wholly dependent on "the axiom of Archimedes." If instead of this midpoint postulate it is assumed that the sum of two sides of every triangle is greater than the third side and that every two right angles are equal, then Hilbertian three-dimensional geometry follows (with the same restriction as above). Other alternative sets of postulates are discussed and among other results is obtained the theorem that if to these postulates is added the assumption that the sum of the angles of every triangle is two right angles, then ideal points may be introduced in such a manner that the thus ex-

tended geometry is euclidean though (as has been shown by Dehn) the original might not have been.

3. In this paper the isomorphism between a quaternion and a linear fractional substitution $\begin{pmatrix}\alpha\beta\\ \gamma\delta\end{pmatrix}$ is made use of to develop the existence of finite groups of quaternions. These may be isomorphic to the rotation groups, extended rotation groups, congruence groups, and groups with $\alpha$, $\beta$, $\gamma$, $\delta$ in any abstract field.*

4. The four deep principles of mathematics are taken to be the ensemble, characteristic of arithmetic; the correspondence, characteristic of the theory of functions; generation, characteristic of groups; relation, characteristic of algebras. Relations are of second, third, or higher rank, according as they are between two, three, or more elements. Relations of a given rank are uniform when, if $R_{12\ldots n}$ be the relation, of rank $n$, there are $n$! relations existing: $R_{\alpha\beta\ldots\nu}$; where $\alpha, \beta, \ldots, \nu$ is any permutation of $1, 2, \ldots, n$, and all the relations exist for any given set of $n$ elements $a, b, \ldots, n$. These relations combine in a manner isomorphic with the symmetric substitution group on $n$ letters. These lead in the case of rank $n$ to $n$ sets of $n$ identities corresponding to the combinations $R_{\alpha'\beta'\ldots\nu'}$, $R_{\alpha''\beta''\ldots\nu''}$. Other equations may be added to these, which are called limitations and furnish the defining limitations of particular types of algebras, as, *e. g.*, associative algebras. When different relations, as $R'$, $R''$, $\ldots$, $R^{(m)}$, exist in an algebra, the algebra is of grade $m$ corresponding. We thus have distributivity and other like relations.

5. So far as the determination of the primitive groups of a given degree has been carried out, there occur only two simply transitive primitive groups which are simple groups of composite order. It is the object of Professor Rietz's paper to call attention to a system of such groups. It is proved that there is a group of this type whose degree is congruent to 1 modulo $p$ corresponding to every prime number $p > 11$, when $p$ is itself twice a prime number plus one.

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* Mr. Wedderburn has called my attention to the paper by W. I. Stringham *American Journal*, vol. 4 (1882) pp. 345–358, on "Determination of the finite quaternion groups," where the same results are obtained. J. B. S.

6. The question of the existence of simple groups of even orders less than 2000 has been dealt with by Hölder, Cole, Burnside, and Ling and Miller. The object of Mr. Wilson's paper is to continue the search up to the order 5000. It is shown that between these limits, there can be no simple group of even order other than 2448, 2520, 3168, 3420, 3744 and 4080, and that in the first, fourth, and sixth cases such a group must be holoedrically isomorphic with the linear fractional groups of these orders, and in the second case with the alternating group of degree 7. The question of the existence of simple groups of the remaining two orders is not yet answered.

7. The first paper by Professor Dickson treats of the cyclotomic function $Q_n(x)$ whose roots are the primitive $n$th roots of unity. The usual derivation of the explicit formula for $Q$ is essentially a verification; the present paper obtains $Q$ by means of the recursion formula $Q_n(x) = Q_\nu(x^{p^r}) \div Q_\nu(x^{p^{r-1}})$, where $\nu = n/p^r$, $p^r$ being the highest power of $p$ which divides $n$. For $x$ integral, a common divisor of $Q_n(x)$ and $x^m - 1$ $(m < n)$ must be the greatest prime $p$ dividing $n$, and then $x$ must belong to the exponent $\nu$ modulo $p$. Further, $Q$ is not divisible by $p^2$ except in the trivial case $n = p = 2$, $Q = x + 1$, $x \equiv 3 \pmod 4$. Aside from this exception, and the case $n = 6$, $x = 2$, $Q_n(x)$ and hence $x^n - 1$ has a prime factor not dividing $x^m - 1$ $(m < n)$. The last result yields an immediate proof of the theorem that the commutativity of multiplication is a consequence of the other postulates for a finite field, the double distributive law included. The paper is to appear in the *American Mathematical Monthly* for April, 1905.

8. Professor Loewy calls a linear homogeneous group $G$ completely reducible when, after a suitable linear transformation of variables, the new variables fall into systems such that $G$ permutes amongst themselves the variables of each system according to an irreducible group. It is shown that every linear group which leaves invariant a positive Hermitian form is completely reducible. By a theorem published simultaneously by Loewy and Moore, every finite linear group possesses an invariant positive Hermitian form, and hence is completely reducible, a result due to Frobenius and Maschke. But an infinite linear group need not be completely reducible. Loewy shows that to every linear group $G$ belong $\mu$ completely re-

ducible groups; these are uniquely determined in a definite sequence, if similar groups are not regarded as different. This theorem is in contrast to that on the irreducible Theilgruppen of $G$, which are uniquely determined apart from sequence. The theorems are extended to linear groups in infinite domains. The paper is to appear in the October number of the *Transactions*.

9. Differential parameters of the first order are given in the symbolic representation by

$$(U', \cdots, U^{\lambda}, f', \cdots, f^{n-\lambda})(V', \cdots, V^{\lambda}, f' \cdots, f^{n-\lambda}),$$

where $U', \cdots, U^{\lambda}$ and $V', \cdots, V^{\lambda}$ are functions of $x_1, \cdots, x_n$ and $f', \cdots, f^{n-\lambda}$ equivalent symbols of the differential quadratic quantic $ds^2 = \sum_{i,k=i}^{n} a_{ik} dx_i dx_k$. Professor Maschke's first paper deals particularly with the case where these differential parameters vanish. For example the following theorems are demonstrated:

In order that every direction of the space $S_{\lambda}$: $U' = \text{const.}, \cdots, U^{n-\lambda} = \text{const.}$ may be orthogonal to every direction of the space $S_{n-\lambda}$: $V' = \text{const.}, \cdots, V^{\lambda} = \text{const.}$ at a common point, the equations must be satisfied

$$(V^i, f', \cdots, f^{n-1})(U^k, f', \cdots, f^{n-1}) = \sigma$$

$$(i = 1, \cdots, \lambda; \; k = 1, \cdots, n - \lambda).$$

The condition that each of the two spaces of $\lambda$ dimensions

$$U' = \text{const.}, \cdots, U^{n-\lambda} = \text{const.};$$

$$V' = \text{const.}, \cdots, V^{n-\lambda} = \text{const.},$$

contain at a common point one direction which is orthogonal to all directions of the others, is

$$(U', \cdots, U^{\lambda}, f', \cdots, f^{n-\lambda})(V', \cdots, V^{\lambda}, f', \cdots, f^{n-\lambda}) = \sigma.$$

A number of theorems of a similar nature have been developed in the paper.

10. In this paper Professor Maschke deduces the explicit expression of an invariant of the general quadratic differential quantic $ds^2 = \sum_{i,k=1}^{n} a_{ik} dx_i dx_k$ which, in the case where the space

of $n$ dimensions whose $ds^2$ is determined by the above quantic lies in an euclidean space of $n + 1$ dimensions, coincides precisely with Kronecker's extension of the gaussian curvature. This invariant, which is given by two entirely different expressions for odd and even values of $n$, might then properly be called the gaussian curvature of the quadratic quantic $ds^2$.

This gaussian curvature is also developed for those spaces of $\lambda$ dimensions which are defined by $n - \lambda$ equations

$$U' = \text{const.}, \cdots, U^{n-\lambda} = \text{const.}$$

Here the gaussian curvature appears as a differential parameter of the second order including the functions $U', \cdots, U^{n-\lambda}$.

11. Volterra was apparently the first to make a systematic study of functions whose arguments are arcs of curves. He derived the important theorem that the so-called first variation of the function is expressible in the form

$$\delta F = \epsilon \int_{x_0}^{x_1} F'\eta(x)dx,$$

where $F$ is the function in question, $F'$ its generalized derivative, and the argument and its variation have the form $y = \phi(x)$ and $y = \phi(x) + \epsilon\eta(x)$ respectively. This formula has many applications but loses much of its effectiveness if the argument is restricted to the form $y = \phi(x)$. The object of Professor Bliss's paper is to simplify the proof, and to extend the theorem to curves in parametric representation.

12. The inverse problem of the calculus of variations in the simplest case is: given a differential equation of the form

$$(1) \qquad y'' = f(x, y, y'),$$

to find the integrals

$$I = \int g(x, y, y')dx,$$

whose extremals are defined by the equation (1). In this form the problem has been solved by Darboux and Hirsch, and the results have been applied by Hamel, among others, in his extremely interesting dissertation "Ueber die Geometrien in denen die Geraden die Kürzesten sind." A study of this paper impresses one, however, with the difficulties which are artificially introduced when the curves considered are restricted to the form $y = \phi(x)$. In Professor Bliss's second paper this restriction is removed, and the results extended to curves in parametric representation.

13. The general problem in Mr. Underhill's paper is to show the invariance of the four necessary and sufficient conditions in the calculus of variations under contact transformation. The conditions are designated by E, L, J, W. The problem is of order $n$ when $n$ derivatives occur under integrand function. The invariance has been shown in the following cases: ($a$) Non-parametric form, (1) E, L, J, W, for extended point transformation, first and second order problems; (2) E, for Lie contact transformation, first order problem; (3) E, L, J, W, for Lie contact transformation, second order problem. ($b$) Parametric form, (1) E, L, J, W, for extended point transformation, first order problem; (2) E, L, W, for extended point transformation, second order problem; (3) L, W, Lie contact transformation, second order problem. E, under $a_2$, $a_3$ is independent of Kürschak in *Mathematische Annalen*, volume 55, where it is shown for problem of the $n$th order. E, L, W, under $b_1$, *cf.* Bolza, Calculus of variations, § 35.

14. In Mr. Graber's paper the cyclide is defined as a fourth order point surface consistent with the equations (1) $b^2 + c^2 + d^2 - ae = 0$ and (2) $\phi\,(a, b, c, d, e) = 0$, where (1) is the second degree equation which remains unchanged by the point transformations of ordinary sphere geometry, and (2) is any equation of the second degree. A classification of cyclidal surfaces on the basis of nodal conics, connectivity with the real surface, and surface deficiency formulæ, is then briefly outlined. The ordinary surfaces of the second degree are special cases of cyclides, and generalizations of value in classification may be obtained by extending the known properties of quadric surfaces to cyclides.

15. The second paper by Professor Dickson is devoted to a generalization of the important results obtained by Maillet (*Annales de Toulouse*, 1904) on the Galois group of the equation defining certain types of geometric configurations and the application to the number of real elements in the configuration. Employing a different method of attack, the paper establishes the generalizations quite simply, in contrast to the elaborate

analysis of Maillet. For the case of the 28 bitangents to a quartic curve, it is shown by group theory that the number of real bitangents must be 4, 8, 16 or 28, in accord with Schlafli's geometric analysis. Maillet's method does not exclude the impossible case of no real bitangents. For the case of the 27 straight lines on a cubic surface, 3, 7, 15 or 27 must be real, a result in accord with that of the geometers. The paper has been offered to the *Annals of Mathematics*.

16. Relations of order and continuity play relatively insignificant rôles among the classical theorems of synthetic projective geometry. These theorems can nearly always be stated in terms of incidence relations. Dr. Veblen has shown that practically the whole field covered by elementary courses in projective geometry can be developed from axioms of incidence together with the assumption that, if two projective ranges have their point of intersection self-corresponding, they are perspective. On the basis of these axioms, an analytic geometry can be developed in which the coordinates are marks of a field. Conversely, an analytic geometry whose coordinates are marks of an arbitrary field satisfies the axioms indicated. A slight modification of the theory is needed for a field with modulus 2. The case of finite fields is discussed in the paper of Dr. Bussey. A synthetic treatment of projective geometry according to this scheme is contained in a set of lecture notes on a course at the University of Chicago, compiled and mimeographed by Messrs. N. J. Lennes and R. L. Börger.

17. On account of the one to one correspondence between the totality of real numbers and the points of a straight line, there is a close connection between algebra and geometry, and consequently an advance in one often carries with it an advance in the other.

If the analogy is to hold when the algebraic domain is finite, there should be, in the realm of abstract geometry, a finite geometry to correspond to the algebra of every finite field. The paper by Dr. Veblen and Dr. Bussey exhibits a finite $n$-dimensional geometry ($n = 0, 1, 2, 3, \ldots$), corresponding to every finite field. The usual names point, straight line, plane, 3-space, . . ., and $n$-space are given to the geometries of 0, 1, 2, 3, . . ., and $n$ dimensions respectively. The finite geometry corresponding to a field of order $s$ is obtained by taking a linear homogeneous equation in $k+1$ variables as the equation of a $(k-1)$-space in $k$-dimensional geometry, the domain for the coefficients and variables being the field of order $s$. These geometries are also studied from the point of view of pure synthetic geometry. A study is made of curves and surfaces of the second order; also of the groups of the geometries. Finally certain derived tactical configurations are obtained among which are triple systems in $t$ elements where $t = 2^k - 1$ or $3^k$.

18. Frobenius and C. S. Peirce have shown that, in the domain of real numbers, the only linear associative algebras every element of which, except zero, possesses an inverse are quaternions and its subalgebras. Mr. Wedderburn shows that the Galois field is the only algebra of this type which possesses a finite number of elements.

19. Mr. Owens's paper is principally concerned with these two subjects: (*a*) The introduction of the plane ideal elements of projective geometry, independently of the parallel axiom, and of continuity. (*b*) A proof, independent of the parallel axiom and of continuity, that a certain form of the Desargues theorem is a sufficient condition that a plane satisfying order axioms may be a part of a projective three-space.

20. In the paper of Dr. Veblen and Mr. Wedderburn, it is pointed out that Hilbert's proof that Pascal's theorem is not deducible from Axioms I, II, and III of his "Grundlagen der Geometrie," is incorrect, namely the algebra of his non-pascalian geometry does not satisfy axiom 16, § 13. A proof of Hilbert's theorem is given by a modification of his geometry and a second proof that Pascal's theorem cannot be deduced from axioms I and III is obtained by means of an analytic geometry based on quaternions. Lastly, a new set of finite configurations, which may be regarded as non-desarguesian finite geometries, is discussed.

21. Dr. Veblen's note calls attention to certain connections between order relations and the existence of square roots in a number system. Among other ways of stating the relation is the following. Define $-1$ as a mark such that $-1+1=0$, a square as a mark for which there exists in the field considered a mark $x$ such that $x.x = a$, and a not-square as a mark

which is not a square; one may add to any set of postulates defining a field these two: ($\alpha$) $-1$ is a not-square. ($\beta$) The sum of two not-squares is a not-square. On defining $a < b$ to mean that $a - b$ is a not-square, the usual propositions about the symbol $<$ follow at once. If a continuity axiom is added, the system of postulates so obtained defines the real number system. This note is to be published in the *Transactions*.

22. The third paper by Professor Dickson deals with the generalization of the concept of hypercomplex number systems and of the precise definition of this generalized concept by means of independent postulates. The elements are $a = (a_1, \cdots, a_n)$, where the $a_i$ are marks of a given field $F$. A system of such elements together with $n^3$ fixed marks $\gamma_{ijk}$ form a number system if the following six postulates hold: (1) if $a$ and $b$ are elements of the system then $(a_1 + b_1, \cdots, a_n + b_n)$ is also an element; (2) the element $0 = (0, \cdots, 0)$ occurs in the system; (3) if 0 occurs, then to any element $a$ corresponds an element $a'$ of the system such that $a_i + a_i' = 0$ $(i = 1, \cdots, n)$; (4) if $a$ and $b$ are any two elements, then $(p_1, \cdots, p_n)$ is an element of the system, where $p_i = \Sigma a_j b_k \gamma_{jki}$; (5) the usual relations between the $\gamma$'s assuring associativity of multiplication; (6) if $\tau_1, \cdots, \tau_n$ are marks of $F$ such that $\tau_1 a_1 + \cdots + \tau_n a_n = 0$ for every $a$, then $\tau_1 = 0, \cdots, \tau_n = 0$. It is shown that $n$ units linearly independent with respect to $F$ can be determined. The paper is to appear in the July number of the *Transactions*.

J. W. YOUNG,
*Secretary pro tem. of the Section.*

EVANSTON, ILL.,
*May* 20, 1905.

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# A GENERAL THEOREM ON ALGEBRAIC NUMBERS.

BY PROFESSOR L. E. DICKSON.*

(Read before the American Mathematical Society, December 29, 1904.)

1. LET $r_1, \cdots, r_n$ belong to a field $F$ and let

$$(1) \qquad \rho^n = \sum_{i=1}^{n} r_i \rho^{n-i}$$

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