THE APRIL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and twenty-third regular meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York City, on Saturday, April 29, 1905. The meeting extended through the usual morning and afternoon sessions. The attendance included the following thirty-eight members of the Society:

Professor D. P. Bartlett, Professor E. W. Brown, Professor F. N. Cole, Miss E. B. Cowley, Miss L. D. Cummings, Dr. D. R. Curtiss, Dr. W. S. Dennett, Dr. L. P. Eisenhart, Dr. William Findlay, Professor H. B. Fine, Dr. A. S. Gale, Professor L. I. Hewes, Mr. E. A. Hook, Mr. S. A. Joffe, Dr. Edward Kasner, Dr. O. D. Kellogg, Mr. E. H. Koch, Dr. G. H. Ling, Mr. L. L. Locke, Professor E. O. Lovett, Mr. R. B. McClenon, Professor H. P. Manning, Dr. Max Mason, Mr. J. C. Morehead, Dr. L. I. Neikirk, Professor W. F. Osgood, Professor James Pierpont, Miss Virginia Ragsdale, Mr. R. G. D. Richardson, Miss I. M. Schottenfels, Dr. Arthur Schultze, Professor Charlotte A. Scott, Professor D. E. Smith, Professor P. F. Smith, Professor H. D. Thompson, Miss M. E. Trueblood, Professor E. J. Wilczynski, Mr. J. E. Wright.

The President of the Society, Professor W. F. Osgood, occupied the chair, being relieved by Vice-President E. W. Brown and the Secretary. The Council announced the election of the following persons to membership in the Society: Mr. J. H. Grace, Peterhouse, Cambridge, England; Mr. H. B. Leonard, University of Chicago; Mr. R. B. McClenon, Yale University; Mr. W. S. Monroe, Columbia Normal Academy, Columbia, Mo.; Mr. J. C. Morehead, Yale University; Professor Henri Poincaré, University of Paris; Mr. R. G. D. Richardson, Yale University; Miss S. F. Richardson, Vassar College; Mr. F. R. Sharpe, Cornell University; Miss M. S. Walker, University of Missouri. Six applications for membership in the Society were received.

The following papers were read at this meeting:

(1) Dr. ARTHUR SCHULTZE: "Graphic solution of quadratics, cubics and biquadratics."

(2) Dr. MAX MASON: "On the derivation of the differential equation of the calculus of variations."

(3) Dr. D. R. CURTISS: "Theorems converse to Riemann's on linear differential equations."

(4) Miss VIRGINIA RAGSDALE: "On the arrangement of the real branches of plane algebraic curves."

(5) Mr. J. C. MOREHEAD: "Numbers of the form $2^k q + 1$ and Fermat's numbers."

(6) Professor E. B. VAN VLECK: "Some theorems of pointwise discontinuous functions; supplementary note."

(7) Professor JAMES PIERPONT: "Inversion of double infinite integrals."

(8) Professor JAMES PIERPONT: "Multiple integrals (second paper)."

(9) Mr. R. B. MCCLENON: "On simple integrals with variable limits."

(10) Professor E. O. LOVETT: "On a problem including that of several bodies and admitting of an additional integral."

(11) Professor M. B. PORTER: "Concerning Green's theorem and the Cauchy-Riemann differential equations."

(12) Professor M. B. PORTER: "Concerning series of analytic functions."

(13) Mr. J. E. WRIGHT: "Differential invariants of space."

(14) Dr. EDWARD KASNER: "On the trajectories produced by an arbitrary field of force."

(15) Dr. E. B. WILSON: "Sur le groupe qui laisse invariant l'aire gauche."

(16) Professor E. J. WILCZYNSKI: "Projective differential geometry."

(17) Miss I. M. SCHOTTENFELS: "On the simple groups of order $8!/2$" (preliminary communication).

(18) Miss I. M. SCHOTTENFELS: "Certain trigonometric formulæ for the quantity $x + \epsilon y$, where $\epsilon^2 = 0$."

(19) Dr. EDWARD KASNER: "A theorem concerning partial derivatives of the second order, with applications."

(20) Mr. J. E. WRIGHT: "On differential invariants."

(21) Dr. L. P. EISENHART: "Surfaces of constant curvature and their transformations."

(22) Professor L. E. DICKSON: "On the class of the substitutions of various linear groups."

(23) Professor JOSIAH ROYCE: "The fundamental relations of logical and geometrical theory."

Professor Royce's paper was communicated to the Society through Professor Moore. In the absence of the authors, Professor Porter's papers were presented by Dr. Curtiss and Professor Pierpont, and Dr. Wilson's paper by Dr. Gale. The papers of Professor Lovett, Dr. Eisenhart, Professor Dickson, Professor Royce, Dr. Kasner's second paper and Mr. Wright's second paper were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. The usual method of solving equations graphically requires the plotting of a curve for each individual equation — a method too cumbersome to be of great practical value. Dr. Schultze's paper shows that there are certain classes of equations which can be solved by the construction of one curve for each class, and the use of ruler and compasses. The method employed consists in reducing an equation involving one unknown quantity to a system of two simultaneous equations. One of these equations is identical for all equations of the same type, while the other is the equation of a straight line or a circle. By this method all quadratics can be solved graphically by means of the parabola $y = x^2$ and straight lines, or the hyperbola $y = 1/x$ and straight lines; cubics by the cubic parabola $y = x^3$ and straight lines; biquadratics by a parabola and circles. While the main application of the method consists in solving equations of the fourth and lower degrees, it can be employed for certain other equations, for example, Kepler's equation $x - \epsilon \sin x = c$.

2. In Dr. Mason's paper the differential equation which arises from the vanishing of the first variation of multiple integrals was derived by a method analogous to that used by Du Bois Reymond in the case of simple integrals, without assuming the existence of derivatives of the dependent variable higher than those which occur in the integral.

3. In the unfinished paper "Zwei allgemeine Sätze" etc., numbered XXI in his collected works, Riemann develops the following two properties of $n + 1$ linear families made up of the solutions of $n + 1$ homogeneous linear differential equations of the nth order having the same group: 1) All n-rowed determinants of corresponding bases of these families undergo the same substitutions, which are merely multiplicative. 2) There exists a linear relation with single-valued coefficients be-

tween corresponding branches. In Dr. Curtiss's paper theorems converse to these are investigated, criteria thus being obtained that systems of families should have the same group. These results are then illustrated by applications to equations of the second order.

4. The only processes by which curves with the maximum number of circuits have been obtained are the two processes employed by Harnack and Hilbert. Miss Ragsdale's paper shows that on non-singular curves of even order derived by these two processes the circuits conform to a noteworthy law of arrangement. If an *internal* oval of a curve $u = 0$ be defined as an oval which cuts off in the midst of a region where u is negative a region where u is positive, and if an *external* oval be similarly defined, this law can be stated as follows: The number of internal ovals on a non-singular $2n$-ic can not be less than $\frac{1}{2}(n-1)(n-2)$ if the maximum number of circuits is present; and the number of external ovals can not exceed $n^2 + \frac{1}{2}(n-1)(n-2)$ whatever the number of circuits.

Though the proof given applies only to curves derived by these processes, it appears almost certain that the law, as stated, is perfectly general.

5. The first part of Mr. Morehead's paper gives a method and a description of a table for determining expeditiously the primes and the prime factors of the non-primes of the form $2^k q + 1$, k and q being positive integers. The method and the formation of the table are based on several theorems on the form q must have in order that $2^k q + 1 \equiv 0$, mod p, p being an odd prime. With a slight modification these theorems apply equally to the form $2^k q - 1$.

The second part of the paper applies these results to the investigation of possible factors of Fermat's numbers $F_n = 2^{2^n} + 1$, which factors must have the form $2^{n+2} \cdot q + 1$. The net result is that, in addition to the known factors of F_5, F_6, F_9, F_{11}, F_{12}, F_{18}, F_{23}, F_{36}, F_{38}, no more factors under 524,288,000 exist for F_n, $n > 16$, as against the limit 100,000,000 given by Cunningham and Western in the *Proceedings of the London Mathematical Society*, series 2, volume 1 (1904), page 175.

6. Professor Van Vleck's note, supplementary to his October paper on pointwise discontinuous functions, consisted of an

announcement of the two following theorems: 1) If $f(x, y)$ is continuous with respect to x when y is constant, and with respect to y when x is constant, in a field bounded by $x = a$, $x = b$, $y = c$, $y = d$, then

$$F(y) \equiv \int_a^b f(x, y)dx$$

is at most a pointwise discontinuous function for $c \leqq y \leqq d$. 2) If, furthermore, $\partial F/\partial y$ exists in the same field and is continuous with respect to x and y separately, then there is a set of points everywhere dense in (c, d) at which $F(y)$ has a derivative and at which, moreover, the value of $F'(y)$ can be obtained by differentiation of $f(x, y)$ with respect to y under the integral sign.

7. By extending results obtained by Vallée-Poussin in the prize memoir (Brussels Academy, volume 16, 1891–2), Professor Pierpont arrived at the following theorem whose generalization he reserved for another occasion:

Let $f(x, y)$ have no points of infinite discontinuity except on a finite number of lines parallel to the X and Y axes. Let

$$\int_a^\infty f(x, y)dx, \qquad \int_a^\infty f(x, y)dy$$

be in general uniformly convergent in any finite intervals (a, b), (a, β) respectively. Let either

$$\int_a^\infty dy \int_a^x f\,dx \quad \text{or} \quad \int_a^\infty dx \int_a^y f\,dy$$

be uniformly convergent in (a, ∞), (α, ∞) respectively. Then both

$$\int_a^\infty dy \int_a^\infty f\,dx, \qquad \int_a^\infty dx \int_a^\infty f\,dy$$

exist and are equal.

8. Professor Pierpont's second paper related to the reduction of m-tuple proper integrals to iterated integrals, also to the change of variables in such integrals. The only *general* and rigorous treatment of these questions is that of Jordan, for the case of $m = 2$. His methods do not seem to admit of simple general-

ization for higher cases. By an entirely different principle Professor Pierpont establishes the fundamental relation of the type

$$\underline{\int}_{\mathfrak{A}} f\,dx_1\cdots dx_m \leq \underline{\int}_{\mathfrak{x}_m} dx_m \underline{\int}_{\mathfrak{P}_m} f\,dx_1\cdots dx_{m-1}$$

$$\leq \overline{\int}_{\mathfrak{x}_m} dx_m \underline{\int}_{\mathfrak{P}_m} f\,dx_1\cdots dx_{m-1} \leq \overline{\int}_{\mathfrak{A}} f\,dx_1\cdots dx_m$$

and the relation

$$\underline{\int}_{\mathfrak{A}} f\,dx_1\cdots dx_m = \underline{\int}_{\mathfrak{B}} f\left|\frac{\partial(x_1\cdots x_m)}{\partial(u_1\cdots u_m)}\right| du_1\cdots du_m,$$

on the assumption that f is limited in the field $\mathfrak{A}$, concerning whose boundary no restrictions are made except that it is discrete.

9. In Mr. McClenon's paper functions defined by the definite integral

$$J(y) = \int_{\phi(y)}^{\psi(y)} f(x, y)dx$$

are considered. Various conditions for the continuity and differentiability of such functions are established ; and in particular it is proved that the formula

$$(1)\quad \frac{dJ}{dy} = \int_{\phi(y)}^{\psi(y)} f'_y(x, y)dx + f\,[\psi(y), y]\,\psi'(y) - f\,[\phi(y), y]\,\phi'(y),$$

$\phi(y)$ and $\psi(y)$ being one-valued and differentiable, holds true when $f(x, y)$ and $f'_y(x, y)$ have discontinuities along a finite number of lines parallel to the Y axis, provided the singular integrals for those lines converge uniformly to zero for every y in the interval in question. In the case

$$J(y) = \int_{\phi(y)}^{\infty} f(x, y)dx,$$

if (1) fails to define a function, owing to the non-convergence of the integral

$$\int_{\phi(y)}^{\infty} f'_y(x, y)dx,$$

$J(y)$ may still be differentiable, under conditions analogous to those of Vallée-Poussin for constant limits, viz.,

$$\int_{\phi(y)}^{X} f_y'(x, y)dx = g(X, y) + \int_{\phi(y)}^{X} h(x, y)dx$$

where $\int_{\phi(y)}^{\infty} h(x, y)dx$ is uniformly convergent and

$$\int_{b}^{y} g(X, y)dy = 0$$

as X increases without limit. In fact, under these assumptions,

$$\frac{dJ}{dy} = \int_{\phi(y)}^{\infty} h(x, y)dx - f\,[\phi(y), y]\,\phi'(y).$$

The result is that a number of the most important theorems concerning integrals with fixed limits of integration are shown to be valid in the more general case of variable end points.

10. In the problem of three bodies Bertrand introduced certain quadratic functions of the coördinates of the bodies and of quantities proportional to the projections of the velocities on the axes of coördinates. Bour showed that Bertrand's variables satisfy a certain system S of ordinary differential equations of the first order, and pointed out that the problem of three bodies may be considered as a particular solution of a more general problem whose equations are S and of which a certain integral D is known. It is the object of Professor Lovett's note to carry out the extension of these results to the case of any number of bodies. The paper will appear in the *Transactions*.

11. Professor Porter's first paper gave an elementary and simple proof of the theorem

$$\int_{C} f\,dy = \int_{R}\int f'_x\,dxdy,$$

if 1) $f(x)$ is continuous in T; 2) f'_x is limited and integrable in T, in Riemann's sense; 3) f'_x exist in T except perhaps for a point set of (two dimensional) content zero. The case when f'_x becomes infinite in T was also considered, and it was pointed out that if a pair of functions u and v satisfying the Cauchy-Riemann differential equations be such that their first partial derivatives with respect to x and y satisfy conditions 2) and 3), $u + vi$ will be analytic in T, so that it is not necessary to suppose that $f'(z) = (u + iv)'$ exist throughout T in demonstrating Cauchy's integral formula.

12. Professor Porter's second paper was concerned with the proof of the following theorem: If

$$\sum_1^n f_n(z) \qquad (n = 1, 2, \cdots),$$

where $f_i(z)$ is analytic in a given region γ, is *limited* in γ, then

$$\lim_{n=\infty} \sum_1^n f_n(z)$$

will define a set of functions analytic in γ whose number is at least as great as the number of limiting points of

$$\sum_1^n f_n(z_0) \qquad (n = 1, 2, \cdots),$$

where z_0 lies in γ. From this theorem was deduced the conclusion that, if

$$\lim_{n=\infty} \sum_1^n f_n(z)$$

converged for a point set one at least of whose limiting points is in γ, then

$$\sum_1^\infty f_n(z)$$

converges throughout any region of γ to a unique analytic function. This paper will appear in the *Annals of Mathematics.*

13. In Mr. Wright's first paper a functionally independent complete system of differential invariants is obtained as the set of algebraic invariants of a system of m-ary forms, where the space is of m dimensions. Those which appertain to a given manifold of dimensions r are next obtained as the algebraic invariants of a system of r-ary forms. From these are deduced the general forms which determine the differential parameters of a

quadratic differential form. By the use of another method there is obtained a complete system of invariants and parameters of any order for any number of differential forms of any degrees whatever.

14. The equations of motion of a particle in a plane acted upon by a force depending only on position are of the form $\ddot{x} = \phi(x, y)$, $\ddot{y} = \psi(x, y)$, where the dots indicate differentiation with respect to the time. There are ∞^3 possible trajectories, whose differential equation is of the type $y''' = Py'' + Qy''^2$, where P and Q involve x, y, y'. Dr. Kasner discusses the properties of such triply infinite systems of curves. For a given initial speed v_0, there are ∞^1 trajectories through each point. Their centers of curvature are situated on a straight line. There thus arises a correspondence between the points and the straight lines of the plane. If the correspondence is given (arbitrarily), the force is completely defined except in intensity. When the field is lamellar, that is, when the force is conservative, the trajectories are related to the conformal representation of geodesics; when the field is solenoidal, they are related to isogonal trajectories. The case where the correspondence reduces to a reciprocal transformation is discussed in detail.

In the case of central forces, the trajectories fall into ∞^1 doubly infinite systems, one for each value of the sectorial velocity. The only cases where these systems are linear arise when the law of force is of the form $ar^{-2} + br^{-3}$. In the general case, the trajectories through a given point corresponding to a given sectorial velocity have their centers of curvature on a rational cubic curve.

The properties stated are essentially kinematic, since they involve the idea of velocity. Among the purely geometric results, the following is noteworthy. Through any period there pass ∞^1 trajectories in an assigned direction; each of these determines a unique osculating parabola; the foci of these parabolas lie on a circle. Every field of force thus gives rise to a correspondence between the triply infinite manifolds, lineal elements and circles.

In conclusion two general problems of interest were indicated: First, the determination of all forces whose trajectories constitute a linear system; second, the determination of all forces whose trajectories admit continuous groups. The familiar central forces (elastic law and Newton's law) possess both of these properties.

15. In the *Jahresbericht der deutschen Mathematiker-Vereinigung*, volume 12 (1903), Dr. Wilson gave a generalized definition of area and volume independent of the conception of length and especially suitable for use in projective geometry. This was applied to a discussion of the projective group in the plane with particular reference to transformations resoluble into two projective reflections. See *Annals of Mathematics*, volume 5 (1903). M. Fréchet has generalized the notion of area so as to obtain the area of a twisted curve in space. See *Nouvelles Annales de Mathématiques*, June, 1904. In the present note Dr. Wilson determines the group that leaves area invariant in space. This group is identical with that which leaves length invariant. The analysis is carried out with particular ease by using the methods introduced into multiple algebra by J. Willard Gibbs.

16. Professor Wilczynski showed that projective differential geometry is closely bound up with the theory of invariants of linear differential equations, or of systems of such equations, ordinary or partial. He gave a brief outline of the theory of plane and space curves, and of ruled surfaces as developed by him under the auspices of the Carnegie Institution. A systematic account of this theory will be found in a volume which is soon to appear, published by Teubner. He also indicated how a general theory of surfaces may be constructed, governed by the same general ideas.

17. In Miss Schottenfels's first paper, it is shown that all simple groups of order 20160 can be expressed as substitution groups upon 35 or a smaller number of letters.

18. Miss Schottenfels's second paper is devoted to a development of the analogues of the ordinary trigonometric formulas for the hypercomplex variable $x + \epsilon y$ where $\epsilon^2 = 0$.

19. The theorem of Dr. Kasner's second paper concerns the condition which must be satisfied by three functions A, B, C of two variables x, y if it is possible to find a function f so that $A : B : C = f_{xx} : f_{xy} : f_{yy}$. Application is made to the inverse problem of the calculus of variations, to a certain class of point transformations, and to the asymptotic lines of surfaces. For example, it is shown that when the asymptotic lines are projected orthogonally upon a plane, the result is an isothermal net whenever it is an orthogonal net.

21. Dr. Eisenhart considers a surface of constant positive curvature referred to an isothermal-conjugate system of lines. When the parameters are suitably chosen, the two fundamental quadratic forms and the equations of condition defining their coefficients can be so written as to disclose the existence of a second surface with the same second quadratic form and total curvature whose linear element is obtained by interchanging the first and third coefficients of the linear element of the given surface and changing the sign of the second coefficient. When the given isothermal-conjugate system is composed of the lines of curvature, which is possible for every surface of constant curvature, this transformation is the one first considered by Hatzidakis and called by Bianchi the Hatzidakis transformation. It is shown that the transformation for any isothermal-conjugate system gives the same surface as the Hatzidakis transformation applied to the lines of curvature. In the foregoing transformation the lines of curvature correspond. Another transformation is found which enables one to derive from a given surface with constant positive curvature a second surface with the same total curvature and second quadratic form, the lines of curvature on the one surface corresponding to the characteristic conjugate lines on the other. The foregoing results are applied to surfaces of revolution with constant positive curvature.

22. Professor Dickson's paper appeared in the May number of the BULLETIN.

23. The paper of Professor Royce is devoted to a restatement and development of the theory of "the relation between the logical theory of classes and geometrical theory of points," which Mr. A. B. Kempe outlined in the *Proceedings of the London Mathematical Society* in 1890. As Mr. Kempe left the statement of the theory in question, it was an account of a development of the principles of projective geometry as a special instance of the laws which are at the basis of the algebra of logic. In the present restatement, the principles of the algebra of logic themselves receive a new formulation differing both from Kempe's and from the various systems of postulates for the algebra of logic recently expounded by Dr. Huntington, in the *Transactions*. A fundamental relation, the "O-relation," is defined by postulates. A system of entities that may or may not stand to one another in O-relations is described.