of position in the Riemann surface. The interest centers in the point systems at which the theta function vanishes. To complete the theory in certain respects, in which it is still open to criticism on account of the possibility of the occurrence of point systems of special character, is the chief aim of this part of the paper. This, together with the critical remarks and illustrative examples collected into an appendix of four pages at the end of the work, form a new and useful contribution to the Riemann theory.

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MATHEMATICAL CRYSTALLOGRAPHY.

Mathematical Crystallography and the Theory of Groups of Movements. By HAROLD HILTON Oxford, The Clarendon Press, 1903. 262 pp.

THE problem of crystallography is to establish a correspondence between chemical composition or certain abstract aspects of it and geometric form in crystals.

Setting aside the work, usually assigned to the mineralogist, of reducing the actual forms of crystals to idealized representatives, and the chemist's problem, of establishing the molecular configuration, and showing why given molecules are more likely to be piled in one way than another, the subject matter of mathematical crystallography is the formal problem of showing a correspondence between the idealized forms of the mineralogist and the geometrically possible methods of piling, that is of filling space of three dimensions with interpenetrant homogeneous assemblages. Mr. Hilton's work undertakes to set this out as it has been done by Bravais, Sohncke, Schoenflies and others, and also to supply such geometric material as may be pertinent.

Dealing first with the main issue the definition of subject matter (page 11) amounts to this: The idealized forms of natural crystals are polyhedra with rational indices; more explicitly polyhedra whose faces are parallel to the set of planes

$$\frac{hx}{a} + \frac{ky}{b} + \frac{lz}{c} = 0,$$

where a, b, c are any real positive quantities and h, k, l, the "indices," are integers, usually small.

This is a sufficient definition and does not need combination (page 11) with the law of constancy of angles, which is a consequence unless $a : b : c$ vary, as in calcite with the temperature.

The formulas given (page 20) for calculation of indices involve four independent angular measurements in a manner invariant under this expansion.

The line of argument is then briefly:

1. Each of the rational index polyhedra (R) is invariant under some point-group (P) of a set of 32. Point-groups are rotation groups or extended rotation groups.

2. There exists a set of 14 lattices, or homogeneous assemblages of points (A), each one invariant under a translation group T containing 3 independent finite translations and no infinitesimal translation.

3. Each lattice A is invariant under some P and only under point groups of the set P. There exist 230 space-groups (S') which contain some T as a subgroup and some P as a maximum subgroup of the class.

4. Every system of interpenetrant homogeneous assemblages (M) is invariant under some S and only under space-groups of the set S.

The crystallographer and chemist are then left with the problem of determining, for a given crystal of geometric form R and given chemical and physical properties, which of the sets S and M shall be assigned to it.

To establish the invariance of R under P it is proved (Chapter III) that symmetry axes only of orders 2, 3, 4, 6, occur in R. There is trouble and apparently avoidable trouble with the proof. As a lemma it is proved (page 40) that a symmetry axis is perpendicular to a possible (*i. e.*, rational index) face. This does not hold for three-fold axes as is carefully pointed out. Still a corollary of this lemma is used (§ 19) to prove $\cos 2\pi/n$ rational, whence $2 \cos 2\pi/n$ must be integral. The conclusion $n = 2, 3, 4$ or 6 ignores the formal solution $n = \infty$, and of course the chain of reasoning for $n = 3$ is hopeless. The exclusion of 5, 7, 8, etc., and the admission of 2, 3, 4, 6, which is the object of the discussion, can be accomplished however, and the appeal to the structure theory (page 39) is irrelevant.

The enumeration of (P) the point groups having only 2-al, 3-al, 4-al, and 6-al symmetry axes is given in Chapters V and VI. After a short introduction to the theory of groups the method is direct and involves the metrical element in the way of spherical areas. Hence Euler's theorem, one of analysis situs applicable to a much wider class of objects than those discussed, is said rather strangely to be *deduced* (page 69).

Two proofs somewhat similar are given (§ 4 and § 5). In the former the angle between axes is assumed acute (page 56). This excludes the tetrahedron gronp as generated by 3-al rotations, yet the case comes back in the solution of an inequality in integers to be thrown out again on the ground of the obtuse angle. In the second proof the spherical triangle formed by vertices of the inscribed tetrahedron is specifically excluded. The tetrahedron, however, turns up in the table (page 59) to be omitted in the subsequent discussion until its group is reinstalled as a subgroup of the octahedral. As a crystal the tetrahedron is a tetartohedral form, its group being a subgroup ot index 4 under the extended octahedral group, but there seems no reason to insist on this at this stage.

The rotation groups are extended by an operation of the "second sort" in Chapter VI. The diagrams, stereographic projections, are clear and satisfactory.

The general theory of rows, nets and lattices is given in Chapter XIII and the possible symmetry of lattices discussed in the following chapter. This is done clearly and well.

The enumeration of the 230 space-groups necessarily involves a rather lengthy statement (Chapters XVIII to XXIII), and though it might be possible and desirable to arrive at the number and a bare classification with less labor, yet from the point of view of aiding definite work in placing actual crystals in correspondence with the different space-groups the more detailed work should stand. The figures, of course rather symbolic, supply an adequate representation.

In addition to the special subject the book contains an exposition of various pure geometrical theorems connected with it. The theory of stereographic projection claims Chapter I. Here it might be noted that the lemma of § 4 can be proved by symmetry without so direct an appeal to the congruence of triangles. If the student needs this the use of "perpendicular" to describe a relation between non-coplanar lines (page 18) errs in excessive brevity.

The analytic geometry of rotation as treated in Chapter VII uses a notation involving the cosines of angles written explicitly. In Note I to the chapter the author calls attention to the greater neatness of the direction ratio notation but uses only a single subscript. Note II introduces a double subscript notation but only to prove an elementary determinant identity by the construction of a rather unwieldy polygon in three dimensions.

The discussion of the properties of geometric operations in Chapter XVI makes good use of the group theory and is very satisfactory.

The suggested practical method of finding the resultant of several screws involves the determination of the first and last positions of some "easily ascertained point." Of course, there may be, and in the application to crystallography (where the screw angles are limited to a few aliquot parts of the complete rotation) usually are such points. In general, however, all points are about in the same case.

The exposition of the group theory here required is rather abbreviated but satisfactory, except the definition of "isomorphous" (page 158). This word, usually kept for the chemical connotation, replaces practically the word "isomorphic" as used in group theory as far as it applies to this subject. But the restriction in the definition is hardly necessary.

The proposition (page 159) that "the symmetry of lattice is in general the holohedry of which a point group under which it is invariant is the merohedry" seems to use the words "in general" to mean in a majority of a finite number of cases. There seems to be no general theorem applicable, since the quotient groups of two groups may be related as group and subgroup without the original groups being so related, whenever the denominators in the quotient symbols differ.

The necessary references to the historical and physical side of the subject are satisfactory. The definition of crystalline (page 8) seems to be that usually given for eolotropic. Do there not exist outside the archimedean universe we are here dealing with "bodies" properly called eolotropic and yet not crystalline? The uniformly magnetized bar is not usually classed as crystalline; its geometric ideal has a symmetry axis of infinite order, the case ignored by the author.

The whole subject is one of extreme interest to the physicist and chemist, and the mathematics involved, though rather special, constitute a good field for the exercise of geometric powers.

Thanks are due to the author for his labors in gathering these things together and setting them forth in a connected way.

R. P. BAKER.