THE APRIL MEETING OF THE CHICAGO SECTION.

THE fifteenth regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY was held at Northwestern University, Evanston, Ill., on Saturday, April 2, 1904. The following members were present:

Dr. G. A. Bliss, Professor Oskar Bolza, Professor D. F. Campbell, Professor L. W. Dowling, Mr. E. B. Escott, Professor Arthur G. Hall, Dr. C. N. Haskins, Professor E. R. Hedrick, Professor Thomas F. Holgate, Dr. H. G. Keppel, Mr. N. J. Lennes, Mr. A. C. Lunn, Professor Heinrich Maschke, Professor E. H. Moore, Professor G. W. Myers, Dr. H. L. Rietz, Professor J. B. Shaw, Professor E. B. Skinner, Professor S. E. Slocum, Professor E. J. Townsend, Dr. Oswald Veblen, Professor C. A. Waldo, Professor H. S. White, Mr. J. W. Young, Professor Alexander Ziwet.

The first session opened at ten o'clock, Professor George W. Myers, of the University of Chicago, in the chair. The following papers were read:

(1) Dr. A. E. YOUNG: "Surfaces which have $D = \pm D''$ when referred to isothermal lines."

(2) Dr. G. A. BLISS: "On the sufficient condition for a minimum with respect to one-sided variations in the calculus of variations."

(3) Mr. A. C. LUNN: "Transformations of the equation of plane waves."

(4) Mr. E. B. ESCOTT: "Some new logarithmic series."

(5) Professor J. B. SHAW: "Group algebras defined by the groups whose generating equations are $e_1^a = 1 = e_2^c$, $e_1e_2 = e_2e_1^m$."

(6) Dr. SAUL EPSTEEN and Mr. H. B. LEONARD: "On the definition of reducible linear associative algebras."

(7) Professor J. B. SHAW: "The group determined by a matrix of order n is of order n^3."

(8) Mr. A. C. LUNN: "The fundamental theorems on ordinary differential equations in real variables."

(9) Mr. E. B ESCOTT: "Expression of the square root as a continued fraction."

(10) Dr. G. A. BLISS: "The exterior and interior of a plane curve."

(11) Professor S. E. SLOCUM : "The strength of flat plates, with an application to concrete-steel floor panels."

(12) Dr. SAUL EPSTEEN : "On linear homogeneous difference equations."

(13) Professor W. J. RUSK : "Note on the nth derivative of a determinant whose constituents are functions of a given variable."

(14) Professor L. E. DICKSON : "Application of groups to a complex problem in arrangements."

(15) Professor L. E. DICKSON : "The abstract group G simply isomorphic with the group of linear fractional transformations in a Galois field."

Dr. Epsteen and Mr. Leonard were introduced by Professor Moore. In the absence of the authors the papers by Professor Rusk and Professor Dickson were read by title.

During the noon hour luncheon was served, at which over forty persons were present. This was followed by a discussion of the present state of mathematical study in Europe, participated in by Professor Maschke, Dr. Bliss, Professor Townsend and Professor Hedrick.

Abstracts of the papers presented follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Dr. A. E. Young considered the problem in differential geometry of determining those surfaces which have $D = \pm D''$, when referred to isothermal lines, and when the parameters chosen are such that the square of the linear element takes the form $\lambda(du^2 + dv^2)$, the D's being the well known quantities appearing in the second fundamental form. Considering first the case $D = -D''$, he finds that every surface whose fundamental forms satisfy these conditions, is a minimal surface. Also, all minimal surfaces satisfy these conditions. Finally, minimal surfaces are characterized by the property that when referred to any isothermal system whatsoever, they have $D = -D''$ in their second fundamental form.

Proceeding to the case where $D = D''$, Dr. Young finds, as a solution of the Codazzi and Gauss equations, surfaces whose linear element takes the form

$$ds^2 = \left[\frac{[1 - e^{a(u+v)}]^2}{be^{a(u+v)}}\right]^{\tan^2 w/2}(du^2 + dv^2),$$

where w is the asymptotic angle.

These surfaces, infinite in number, are shown to be peculiarly related to one another and to the catenoid. They all have a constant angle between their asymptotic lines, but no two the same angle.

Lastly, there is a system of surfaces for which $D = D''$ whose linear element takes the form

$$ds^2 = [f(u+v) + \varphi(u-v)]^{(c-1)/c}\,[du^2 + dv^2],$$

the possible values of c and the form of the functions f, φ being determined by the Gauss equation.

The paper closes with the complete determination of these latter surfaces.

2. In the usual problem of the calculus of variations, a curve E, $y = y(x)$, is sought which joins two fixed points p and q and gives to an integral

$$I = \int f(x, y, y')dx$$

a smaller value than any other curve C joining p and q. In Dr. Bliss's first paper the curves C are supposed to lie on one side only of the curve E, and a set of sufficient conditions for a minimum under those circumstances is derived. The result is of importance when the curves along which I is taken are restricted to lie within a region R of the (x, y)-plane. In that case an arc of the solution E may coincide with the boundary of R, and the only allowable variations of such an arc are therefore all on one side.

3. Mr. Lunn's note gave the form of the function group defined by the condition that the equation of plane waves be invariant. The analogy with the theory of conformal transformation was pointed out, and the method applied to the construction of classes of possible vibration modes whose computation by Fourier's series would be comparatively tedious.

4. In Mr. Escott's paper, the author starts with the well-known series

$$\log\frac{1+y}{1-y} = 2(y + \tfrac{1}{3}y^3 + \tfrac{1}{5}y^5 + \cdots).$$

If we replace y by $C(a_0x^n + a_1x^{n-1}\cdots + a_n)^{-1}$ the first member takes the form of a fraction, the numerator and denominator being polynomials of the nth degree and differing only in their absolute terms. The problem is to determine the coefficients so that the numerator and denominator may have rational, linear factors. The problem is completely solved for $n = 1, 2, 3$. For $n = 4, 5, 6$, an indefinite number of solutions may be found, some of which have been given before. For $n = 10$ four solutions have been discovered, two of the factors however being of the second degree. These are believed to be new. It is shown how the computation of logarithmic tables might be considerably abbreviated by the use of these series.

5. In his first paper, Professor Shaw deduces the conditions which groups of the class defined must satisfy, and develops the general form of corresponding group algebras. From these forms the characters of all operators of the group may be found easily.

6. A linear associative algebra $E_i \equiv E_j E_k \equiv e_1 \cdots e_m e_{m+1} \cdots e_n$, the general number being written $X = J + K$, is said to be reducible and it contains a modulus when the following seven conditions are fulfilled: A) associativity, *i. e.*, $(X_1 X_2) X_3 = X_1 (X_2 X_3)$; C_1) $J_1 J_2 = J_3$; C_2) $K_1 K_2 = K_3$; C_{jk}) $JK = 0$; C_{kj}) $KJ = 0$; C_r) right hand division possible and unique; C_l) left hand division possible and unique. It has been shown (Epsteen, *Transactions*, January, 1904) that the four independent conditions A, C_{jk}, C_{kj}, C_r or C_l suffice for the definition of reducibility.

In the present paper of Dr. Epsteen and Mr. Leonard the above seven conditions are broken up into twenty constituent parts as follows: Since $i = j + k$, A is equivalent to eight conditions $A_1, \cdots, A_8$, which are typified by A_1) $(J_1 J_2) J_3 = J_1 (J_2 J_3)$. The condition C_{jk} is equivalent to the simultaneous fulfillment of C^j_{jk}) $J_1 K_1 \neq J_2$ and C^k_{jk}) $K_3 J_3 \neq K_4$. Similarly C_{kj} is equivalent to C^j_{kj} and C^k_{kj}. Besides the above conditions C_1, C_2, C_r, C_l there are also included C^j_r, C^k_r; C^j_l, C^k_l.

The authors investigate (1) the dependencies among these twenty conditions, (2) all possible modes of defining reducible linear associative algebras, (3) independence proofs of the bodies of conditions which constitute the various definitions of reducibility.

7. In his second paper, Professor Shaw gives a simple proof of the theorem given in the title, deducing also the general form of group algebras defined by the group.

8. Mr. Lunn's second paper deals with sets of sufficient conditions for some of the fundamental theorems on ordinary differential equations in real variables. The theorem of continuity of the solutions as functions of parameters was given in the general case of parameters defined on any set with a cluster point, and shown to have as corollary an extension to differtial equations of the term by term integration of infinite series; this was illustrated by examples from the theory of resonance in vibrating systems, and by certain methods of approximation in celestial mechanics. The theorem of differentiability with respect to the parameters was reduced to the previous theorem and the results applied to equations involving holomorphic functions, and to the commutativity of transformations defined by their infinitesimal transformations. The point transformations defined by a system of differential equations were considered with respect to the maintenance of properties of sets of points, such as continuity, tangency, measurability, with application to the theorems of hydrodynamics.

9. Mr. Escott's second paper considers the relation between a number N and the partial quotients obtained in expressing $\sqrt{N}$ as a continued fraction. A number of special forms of N were given by Euler, Degen, Stern, and others, for which the continued fraction could be written down at once. In this paper a general formula is given which includes all cases. These formulas are expressed by means of Euler's function $[a, b, \cdots m, n]$ (Muir's continuant).

Among the results obtained is this theorem: If $N = 4n + 3$ is a prime number, the continued fraction has a single middle term, a or $a - 1$ according as a is odd or even, a being the greatest integer in $\sqrt{N}$.

10. The theorem that a continuous closed curve divides the plane into an interior and an exterior has been discussed by Jordan, Schoenflies, and in a recent number of the BULLETIN by Ames. The two former assume the theorem for polygons and then extend it to apply to more general curves. The proof of Ames applies to all the so-called regular curves, including polygons. In Dr. Bliss's second paper the theorem is proved by a different method for another class of curves which do not necessarily have a tangent. Among them are included all the curves considered by Schonflies, all curves consisting of a finite

number of analytic pieces, and all the regular curves for which a direction exists parallel to only a finite number of tangents.

11. In Dr. Slocum's paper a rational theory of stress distribution in flat, rectangular plates, fixed at the edges, is developed, and the results obtained are applied to the case of a concrete-steel floor panel in actual use.

To find what proportion of the load at any point of a horizontal plate is carried by each pair of opposite edges, the plate is supposed to be divided by vertical planes into narrow strips parallel to the edges. If we consider two such mutually perpendicular strips through any point of the plate and bearing a unit load at their intersection, the amount of this load transmitted by each strip to its respective abutments is inversely proportional to the deflection of the strip under such load. In this way the equation giving the distribution of load along any strip is obtained and found to be a septic, and from this the position of maximum and minimum load is determined.

In order to simplify the problem sufficiently for practical use, it is then shown that throughout the region under discussion the septic may be replaced by a parabola without involving an error greater than .001 of the total load. Since the maximum bending stress occurs at the edges of the plate, in finding this stress it is allowable to consider any particular strip as an independent beam and analyze it in the usual way. Under parabolic loading the distribution of stress in a strip of unit width is thus determined, and a simple formula for maximum stress deduced.

The case in which the supports are flexible is then taken up. This condition is usually present in the structural use of flat plates, such, for instance, as a concrete-steel floor panel monolithic with its supporting girders. After showing that this problem is indeterminate, an approximate solution is given and a limiting value of the maximum stress determined. The results of the preceding analysis are then applied to the calculation of the maximum stresses in a concrete-steel floor panel of recent design.

12. In the *Comptes Rendus* of October 27, 1903, Guldberg shows that to a linear homogeneous difference equation

$$E: \qquad y_{x+n} + p_x^{(1)} y_{x+n-1} + \cdots + p_x^{(n)} y_x = 0$$

there corresponds a linear homogeneous group

$$\Gamma: \qquad \bar{y}_x^{(i)} = \sum_{j=1}^{n} a_{ij} y_x^{(j)} \qquad (i = 1, \cdots, n)$$

which has the following double property: Every rational function of

$$x, \quad y_x^{(1)}, \quad \cdots \quad y_x^{(n)},$$

and their successive values (the y_x forming a fundamental system), which is expressible rationally as a function of x, remains invariant when the y's are transformed by Γ, and conversely.

Dr. Epsteen points out in his second paper that the above double theorem relates to *formal* and not to *numerical* invariance; he indicates the importance of specifying definitely a domain of rationality and then shows that the group of rationality G of E is a mixed group of r systems of linear homogeneous transformations

$$G: \qquad \bar{y}^{(i,j)} = \sum_{k=1}^{n} a_{ikj} y_x^{(k,j)} \qquad \left(\begin{smallmatrix} i=1, \cdots, n \\ j=0, 1 \cdots, v-1 \end{smallmatrix}\right).$$

By the adjunction of the solutions of an algebraic equation with rational coefficients to the domain of rationality, the group of rationality G is reduced to a continuous group G_1. When G is the identical transformation the difference equation is algebraically integrable.

13. Professor Rusk obtained a general expression for the nth derivative of a determinant.

14. The first paper by Professor Dickson determines all the group solutions of the following problem: Required $\frac{1}{2}(m-1)(m-2)$ arrangements of m persons at a round table such that no one shall be twice between the same two companions. Give to the arrangements the notation $AB \ldots M$, the first letter being always A. We require that the substitutions by which they are obtained from a standard order $ABC \ldots K$, together with their left-hand products by T which replaces $ABC \ldots K$ by $AK \ldots CB$, shall form a doubly transitive group G. Since G is of order $(m-1)(m-2)$ on $m-1$ letters, we must have $m-1 = p^n$, p being prime. Furthermore, G must be the group of linear transformations $x' = ax + b$ in the $GF[p^n]$, there being, however, a second group H_{72} for $p^n = 3^2$. For the first non-trivial case $p^n = 2^2$, the only solution is

$$\begin{array}{ccccc} \infty & 0 & 1 & i & i^2, \\ \infty & 0 & i & i^2 & 1, \\ \infty & 0 & i^2 & 1 & i, \end{array} \qquad \begin{array}{ccccc} \infty & 1 & 0 & i^2 & i, \\ \infty & 1 & i & 0 & i^2, \\ \infty & i & 0 & 1 & i^2. \end{array}$$

For $p^n = 5$, the only group solution is one equivalent to that by C. H. Judson in the *American Mathematical Monthly*, 1900, page 72. For $p^n = 7$ there are exactly four group solutions; for $p^n = 9$, there are four. The second group H_{72} actually furnishes solutions.

15. The second paper by Professor Dickson relates to the question of redundancies in Professor Moore's set of relations defining G for the $GF[p^n]$. It is an addition to a paper with the same title in the *Proceedings of the London Mathematical Society*, volume 35 (1902), pages 292–305, which treated the cases for which $p^n \leqq 48$. In the present paper, the theorem of the paper cited is proved, by fortunate devices, to hold for the two new cases $p^n = 3^5$ and $p^n = 5^3$.

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EVANSTON, ILL.,
April 15, 1904.

THE HEINE-BOREL THEOREM.

BY DR. OSWALD VEBLEN.

This note has for its main object a proof that the Heine-Borel theorem is equivalent, as a continuity axiom, to the Dedekind cut proposition.* In §3 the equivalence of the closure of a set with what may be called the H-B property is shown to apply not only to the continuum but to any set of numbers.† This result applies in manifolds of any number of dimensions as well as to linear sets.

1. I. *If every number of an interval AB belongs to at least one segment σ of a set of segments [σ], then there exists a finite sub-*

*The equivalence in question suggested itself to Mr. N. J. Lennes and myself while we were working over some elementary propositions in real function theory.

†This extends a theorem of Dr. W. H. Young to the effect that every closed linear set has the H-B property. Cf. W. H. Young, "Overlapping Intervals." *Proceedings of the London Mathematical Society*, p. 384, vol. 35 (1903).