Substituting this value in (13), we find that the coefficient of the highest power of x in the first member is $-2\alpha^3$. This however cannot vanish on account of the assumption that $\alpha \neq 0$. It follows that there are no functions satisfying condition (9) except when $R = 0$. Therefore

No equation of type III, *that is of the Riccati type proper, can represent an isothermal system.*

The complete answer to the question proposed at the beginning of the paper may now be stated :

If the differential equation of an isothermal system is included in the form $y' = P + Qy + Ry^2$, *then the system must belong to one of four species :* 1°, *a set of parallel straight lines ;* 2°, *a pencil of straight lines ;* 3°, *a system of equilateral hyperbolas with common asymptotes ;* 4°, *a system of logarithmic cosine curves* $y + \log \cos x = const.$, *which may also be written* $e^y \cos x = const.$

COLUMBIA UNIVERSITY,
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ON SOME PROPERTIES OF GROUPS WHOSE ORDERS ARE POWERS OF A PRIME.

BY DR. W. B. FITE.

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IN the *Transactions*, volume 3 (1902), page 334, the writer has shown that if a metabelian group of order p^m, where p is a prime, contains an abelian subgroup of order $p^{m-\alpha}$, the p^α power of every operator is invariant. In the same article, page 349, it was shown that a metabelian group of odd order cannot be a group of cogredient isomorphisms if it has a set of generators such that the order of any one of them is not a divisor of the least common multiple of the orders of all the others. This latter was generalized somewhat in the BULLETIN, 2d series, volume 9 (1902), page 140. It is the purpose of the present article to carry this generalization somewhat further and to show that the first theorem is a special form of a theorem applicable to all groups whose orders are powers of a prime.

Let G be a group of order p^m and class k that contains an abelian subgroup G_1 of order p^{m-1}. If A is any operator of G_1 and B any operator of G not contained in G_1, we have

$$B^{-1}AB = At_1,$$

$$B^{-p}AB^p = At_1^{p}t_2^{\frac{p(p-1)}{2}} \cdots t_i^{\frac{p(p-1)\dots(p-i+1)}{i!}} \cdots t_p = A,$$

where

$$B^{-1}t_iB = t_it_{i+1} \quad (i = 1, 2, \cdots, k-1),$$

since B^p is contained in G_1. Also

$$B^{-p}t_iB^p = t_it_{i+1}^{\frac{p(p-1)}{2}} \cdots t_{i+j}^{\frac{p(p-1)\dots(p-j)}{(j+1)!}} \cdots t_{i+p} = t_i.$$

The exponents of the successive commutators are the binomial coefficients, and $t_j = 1$ when $j > k-1$. By giving to i in the equation above the successive values from $k-2$ to $k-p$ inclusive, we see that

$$t_j^p = 1 \; (j > k-p).$$

And in general $t_j^{p^s} = 1$ if $j > k - 1 - s(p-1)$. Hence $t_1^{p^r} = 1$, if $1 > k - 1 - r(p-1)$ or

$$r(p-1) \geqq k-1.$$

Therefore A^{p^r} is invariant in G, and the p^r power of every operator is invariant.

Conversely, if $s(p-1) < k-1$, the p^s power of every operator is not invariant. For otherwise

$$t_i^{p^{s-j}} = 1,$$

if

$$j(p-1) < i \quad (j = 0, 1, 2, \cdots, s).$$

From this it follows that $t_i = 1$, if $s(p-1) < i$. Hence $t_{k-1} = 1$, and t_{k-2} is invariant in G. That is, the $(k-2)$nd adjoined group is abelian, and G is of class l, where $l < k$. The results thus obtained can be formulated into the

THEOREM: *If a group G of order p^m (p a prime) and class k contains an abelian subgroup of order p^{m-1}, the p^r power of every operator is invariant if $r(p-1) \geqq k-1$. Conversely, if G contains an abelian subgroup of order p^{m-1} and if the p^r power of every operator is invariant, G is of class k, where $k - 1 \leqq r(p-1)$.*

We concern ourselves now with the case in which the abelian subgroup is of order p^{m-a}. Let this subgroup be denoted by G_1 and let G_2 be a subgroup of G of order p^{m-a+1} that contains

G_1, and G_3 a subgroup of order $p^{m-\alpha+2}$ that contains G_2. If B is an operator of G_3 that is not in G_2, $B^{p^{r+1}}$ is invariant in G_3. If A is any operator of G_2 that is not in G_1, A^p is invariant in G_2. Now

$$B^{-1}A^pB = A^pt_1,$$

$$B^{-p}A^pB^p = A^pt_1^p \cdots t_i^{\frac{p(p-1)\cdots(p-i+1)}{i!}} \cdots t_p = A^p,$$

where

$$B^{-1}t_iB = t_it_{i+1} \qquad (i = 1, 2, \cdots, k-1).$$

The commutators t_i are invariant in G_2 since A^p is.

As before, it follows that $t_1^{p^r} = 1$, where $r(p-1) \geqq k-1$. Hence

$$B^{-1}A^{p^{r+1}}B = A^{p^{r+1}}t_1^{p^r} = A^{p^{r+1}},$$

and $A^{p^{r+1}}$ is invariant in G_3. If C is any operator of G_1, then C^{p^r} is invariant in G_2, and hence $C^{p^{2r}}$ is invariant in G_3.

We have therefore shown that the p^{2r} power of every operator of G_3 is invariant. Now by induction and a repetition of the argument just used we reach the

THEOREM: *If a group G of order p^m and class k contains an abelian subgroup of order $p^{m-\alpha}$, the $p^{r\alpha}$ power of every operator is invariant, where $r(p-1) \geqq k-1$.*

Returning now to the case in which the abelian subgroup is of order p^{m-1}, we see that if G_1 is of type $(m_1, m_2, \cdots, m_n)$, where l of the quantities m_i are equal to 1, then $k \leqq l+2$ or $> p$. For the commutators of G are all contained in G_1 and if $k \leqq p$ they are all of order p. Furthermore, the pth power of every operator is invariant. Hence the order of the commutator subgroup of G', the group of cogredient isomorphisms of G, is less than, or equal to, p^l, and G' is of class k', where $k' \leqq l+1$.* Therefore $k \leqq l+2$. In particular, if $l=0$, either G is metabelian, or $k > p$.

Suppose now that G is of order p^m and class k and that B' is the operator of G' that corresponds to B of G.† If B' is of order p^β and A is any operator of G, we have, if the commutator subgroup of G is abelian,

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* *Transactions*, vol. 3 (1902), p. 350.

† *Loc. cit.*, p. 332.

$$B^{-p^\beta} A B^{p^\beta}$$

$$= A t_1^{p^\beta} \cdots t_j^{p^\beta(p^\beta-1)\cdots(p^\beta-j+1)/j!} \cdots t_{k-1}^{p^\beta(p^\beta-1)\cdots(p^\beta-k+2)/(k-1)!}$$

$$= A,$$

$$B^{-p^\beta} t_i B^{p^\beta} = t_i t_{i+1}^{p^\beta} \cdots t_{k-1}^{p^\beta(p^\beta-1)\cdots(p^\beta-k+i+2)/(k-1)!} = t_i,$$

$$(i = 1, 2, \cdots, k-2).$$

Since $t_i^{p^{\beta+j}} = 1$, when $(j+1)(p-1) \geqq k-i$, it follows that $t_1^{p^{\beta+r}} = 1$ when $(r+1)(p-1) \geqq k-1$.

In a similar way it can be seen that

$$t_1^{p^{\alpha+r}} = 1,$$

where p^α is the order of A'. Hence $t_i^{p^{\alpha+r}} = 1$, since

$$B^{-1} t_{i-1}^{p^{\alpha+r}} B = t_{i-1}^{p^{\alpha+r}} t_i^{p^{\alpha+r}} = t_{i-1}^{p^{\alpha+r}} \qquad (i = 2, 3, \cdots, k-1),$$

when

$$t_{i-1}^{p^{\alpha+r}} = 1.$$

But we have just seen that

$$t_1^{p^{\alpha+r}} = 1.$$

Suppose now that $k-s$ is the greatest multiple of $p-1$ less than $k-1$. Then since $G^{(s-1)}$, the $(s-1)$th adjoined group of G, is of class $k-s+1$ and $A_{(s-1)}^{p^\alpha}$ is commutative with every operator of $G^{(s-1)}$, we have $t_{1(s-1)}^{p^{\alpha+r'}} = 1$, where

$$(r'+1)(p-1) = k-s.$$

Obviously $r' < r$. Now

$$B^{-1}_{(s-2)} t_1^{(s-2} B^{(s-2)} = t_1^{(s-2)} t_2^{(s-2)}.$$

But since $t_{1(s-1)}^{p^{\alpha+r'}} = 1$, we have $t_{2(s-2)}^{p^{\alpha+r'}} = 1$.

By continuing this reasoning we get $t_s^{p^{\alpha+r'}} = 1$. Hence $t_i^{p^{\alpha+r'}} = 1$, when $i > s$. More generally, we have in $G^{\{s+l(p-1)-1\}}$ the invariant operator

$$A^{p^\alpha}_{\{s+l(p-1)-1\}}.$$

Hence

$$t_1^{p^{\alpha+r_1}\{s+l(p-1)-1\}} = 1,$$

where $r_1 = r' - l$. Then $t_i^{p^{\alpha+r_1}} = 1$, when $i = s + l(p-1)$. This is equivalent to the condition

$$(r_1 + 1)(p - 1) \geqq k - i.$$

We conclude from this that

$$B^{-p^{\alpha+r}} A B^{p^{\alpha+r}} = A.$$

Therefore if A' is of order p^α, the $p^{\alpha+r}$ power of every operator of G is commutative with A, when

$$(r + 1)(p - 1) \geqq k - 1.$$

From this result follows the

THEOREM : *If a group of order p^m and class k has an abelian commutator subgroup, its group of cogredient isomorphisms cannot have a set of generators such that the order of any one of them is greater than p^r times the order of every one of the others, r being defined by the relation $(r + 1)(p - 1) \geqq k - 1$.*

We add a formula for the nth power of the product of any two operators of a group of the 3d class. Let A and B be these operators. Then if

$$B^{-1}AB = At_1, \quad B^{-1}t_1B = t_1t_2, \quad A^{-1}t_1A = t_1h,$$

we have

$$(BA)^n = B^nA^nt_1^{\frac{n(n-1)}{2!}}t_2^{\frac{n(n-1)(n-2)}{3!}}h^{\frac{n(n-1)(2n-1)}{3!}}.$$

CORNELL UNIVERSITY,
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