or, since $e^{is}ds = dc/i$ and $e^{is} = c$ on the circumference,

$$f(z) = \frac{1}{2\pi i}\int_0^{2\pi} f(c)\,\frac{dc}{c - z}\,.$$

The transition from the case of the circle to any region which can be conformally represented on the circle is easy, as Green's function is transformed into Green's function for the new region and $dc/(c - z)$ differs from the corresponding expression for the new region only by a function which disappears upon integration.

PRINCETON, N. J.,
November 20, 1903.

BAUER'S ALGEBRA.

Vorlesungen über Algebra. Von GUSTAV BAUER. Herausgegeben vom Mathematischen Verein München. Leipzig, B. G. Teubner, 1903. vi + 376 pp.

THIS volume was planned in honor of the 80th birthday of Professor Bauer by the Mathematischer Verein of the students of the university and the technical high school of Munich. It presents in fact, not merely in title, lectures as actually given to students in their first or second year at the university, the course extending over two semesters. The preface is by Karl Doehlemann, who saw the book through the press at the request of the Verein.

Treating a wide range of subjects in a strictly elementary manner with many illustrative examples considered in detail, these lectures are certainly very attractive. If one can overlook the lack of rigor in two or three fundamental matters (discussed in detail below), one must regard the volume as one to be specially commended to beginners.

Part I (105 pages), is entitled "General properties of algebraic equations." The usual elementary theorems on complex quantities are given in 12 pages. In the construction of z^n by a series of similar triangles (page 13), n is restricted to positive integers, whereas the series may be continued in the opposite direction to give the negative integral powers. The mere statement that an elementary geometric construction for $z^{1/n}$ is impossible in general would attract the student more if accompanied by the remark that it involved the division of an angle into n equal parts. Chapter III deals with the properties of an integral rational function $f(z)$ of a real or complex quantity z. Its rth derivative is defined as the coefficient of h^r in the expansion (by use of the binomial theorem) of $f(z+h)$. There are established the continuity of $f(z)$ for every finite z, the identity of $f'(z)$ with the calculus limit, the theorems on the derivative of a sum, product or power, and Euler's theorem on homogeneous functions.

Chapter IV begins with the theorem that every algebraic equation has a root, numerous references being given. The proof, however, lacks rigor,* as it merely shows that when $f(z) \neq 0$ a complex quantity h can be found such that $|f(z+h)| < |f(z)|$. Chapter V presents in an unusually simple and attractive form the subject of the decomposition of rational fractions, with application to Lagrange's interpolation-formula. The 18 pages of Chapter VI on symmetric functions is elementary in character, Waring's formula not being established. It is perhaps unfortunate to use as an illustration of a rather elaborate process an example (page 54) which the alert student will doubtless solve almost by inspection. Thus if α, β, γ are the roots of $f(x) \equiv x^3 + a_1x^2 + a_2x + a_3 = 0$, we have directly

$$(\alpha + \beta)(\beta + \gamma)(\gamma + \alpha) = (-a_1 - \gamma)(-a_1 - \alpha)(-a_1 - \beta)$$
$$= f(-a_1) = a_3 - a_1a_2.$$

Elimination is treated partly in Chapters VII–IX and partly in Chapter XXVIII under determinants. As it stands, § 208 presents the usual incomplete derivation of the resultant of two equations in x by Sylvester's dialytic method. The author apparently does not appreciate the need of proving that the values obtained for the various powers of x are consistent. It would have sufficed to refer to § 210, from which the proof follows.

The statement at the top of page 69 that two equations in two variables have always a definite finite number of common root pairs is subject to the (trivial, to be sure) exception arising when the coefficients of one function are proportional to those of the other. A word on the geometry of the problem would

*As shown by the simple example given by Professor Moritz, "On certain proofs of the fundamental theorem of algebra," *American Mathematical Monthly*, vol. 10 (1903), top of p. 160.

add to the interest of the student. The equality of the degrees of the two resultants (from the elimination of x and y, respectively) would then be expected.

Chapter XI presents the transformations of Tschirnhaus and Jerrard, and discusses the construction of an equation whose roots are specified functions (for example, the differences) of the roots of a given equation.

Part II (pages 106–199) treats of the algebraic solution of equations — Cardan's formula, the irreducible case, Euler's and Lagrange's methods, reciprocal and binomial equations, roots of unity, irreducibility of $(x^p - 1)/(x - 1)$, primitive roots.

Chapter XV is devoted to the Abel-Wantzel theorem that the general equation of degree $\geqq 5$ is algebraically insolvable. There is, however, a painful hiatus in the failure to prove (or notice the need of proof of) Abel's theorem that *every equation which is solvable by radicals can be reduced to a chain of binomial equations* (*of prime degrees*) *whose roots are rational functions of the roots of the given equation.*

In the 13 pages of Chapter XVI, the student is given a first view of Galois's theory of algebraic equations. In the opening sections, a factor of an integral function $f(x)$ is called 'rational when its coefficients are rational functions of $f(x)$.' The author should of course say rational functions with integral coefficients. Several pages later is given an extension of the idea of rational factor: 'The coefficients of the equation may be expressible rationally in terms of quantities $e_1, e_2, \cdots$ considered as known. We may then understand by rational factor a factor whose coefficients likewise are formed rationally from $e_1, e_2, \cdots$.' The author is certainly not fortunate here in his introduction of the idea of a domain of rationality. The latter should have an independent definition, for example as the totality of rational functions with integral coefficients of certain quantities $q_1, q_2, \cdots$. When the problem relates to a given algebraic equation, we naturally take as the initial domain of rationality one including all the coefficients, but may agree to admit also quantities either wholly arbitrary or involving irrationalities not appearing in the coefficients. Without this general conception of a domain of rationality, the idea of enlarging the domain by adjunction would be undesirably restricted.

The rudiments of the theory of numbers are given in the 20 pages of Chapter XVII. The two following chapters deal with abelian and cyclotomic equations, with a detailed discussion of the division of the circumference into 5, 7, 13 and 17 equal parts.

Part III (pages 200–350) presents the numerical solution of equations under the topics: limits for real roots; Descartes's rule of signs *; the theorems of Rolle, Fourier, Sturm; methods of approximation due to Newton and Lagrange, and the simple *Regula falsi;* the theorem of Cauchy for the number of imaginary roots within a given region of the complex plane. The method of solution of numerical equations over which the author is most enthusiastic is that of Graeffe, as improved by the astronomer Encke. The principle is the derivation of a second equation whose roots are such high powers of the roots of the given equation that the powers of the smaller roots are, in relation to those of the larger roots, negligible quantities.

Part IV (pages 257–373) is concerned with the theory and application of determinants. The ordinary rule for the multiplication of two determinants of equal order is established very ingeniously by use of linear transformation. However, one step is far from clear (bottom of page 292), as conceivably there is a literal factor of AB which divides the coefficient of each c_i, in the expansion of the numerator. Thus, we might have

$$y_k = \frac{c_1 a_1^{n-1} b_1^{n-1}}{a_1^n b_1^n} = \frac{c_1 a_2^{n-1} b_2^{n-1}}{a_1 b_1 a_2^{n-1} b_2^{n-1}}.$$

Application is made to elimination, discriminants, quadratic and bilinear forms.

The volume concludes with two rather long notes on continued fractions and Lagrange's formula for the sum of the nth powers of the roots of a quadratic equation.

Among the misprints not noted in the full page of errata are fz in § 15, line 2; $\alpha^2 - 1$ on page 56, line 18; 2 for r in (3), page 80; $1\cdot 2 \cdots r$ for $1, 2, \cdots, r$ on page 97, lines 19 and 23, and similarly $x_1 \cdots x_n$ on pages 98, 99 and elsewhere.

The presswork has the usual excellence of Teubner's books. There is a portrait of Professor Bauer as frontispiece.

L. E. DICKSON.

THE UNIVERSITY OF CHICAGO,
November 10, 1903.

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* In the form: "An equation with real coefficients has as many positive roots as changes of sign, or fewer by an even number."