NOTE ON CAUCHY'S INTEGRAL.

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THE analogy between the formula given by Green for a potential function

$$u(x,\ y) = \frac{1}{2\pi} \int_c u(s) \frac{\partial G}{\partial n}\, ds \tag{1}$$

and Cauchy's integral representation of a complex function

$$f(z) = \frac{1}{2\pi i} \int_c f(c) \frac{dc}{c - z}\, dc \tag{2}$$

has been pointed out;[*] the direct deduction of one from the other may be of interest.

We start with the case where the curve C is a circle of radius 1. Let $z = x + iy = re^{i\vartheta}$ represent the variable point within the circle; let $c = a + ib = \rho e^{is}$ represent a parameter point within or on the circle and $c' = e^{is}/\rho = c/\rho^2$ the reflection of the point c with respect to the circle. Then Green's function for the circle is the real part of $\log [c(z - c')/(z - c)]$, so that if $\Re$ denote "the real part of," the formula (1) may be written

$$u(x,\ y) = \frac{1}{2\pi} \int_0^{2\pi} u(s) \frac{\partial}{\partial n} \Re \log \frac{c(z - c')}{z - c}\, ds$$

Noting however that the real and imaginary parts of the logarithm are conjugate functions,[†] we have, if $v(x, y)$ denote the function conjugate to $u(x, y)$,

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[*] See the article in the Encl. d. Math. Wiss: "Analytische Functionen complexer Grössen" (p. 17), by Professor Osgood, to whose suggestion this note is due.

[†] The fact that the derivatives with respect to the normal of a given curve with a determinate tangent of two conjugate functions G and H are still conjugate functions may be verified as follows. G and H satisfy the equations

$$\partial G/\partial x = \partial H/\partial y, \quad \partial G/\partial y = -\partial H/\partial x. \tag{a}$$

If the direction cosines of the given curve be $\cos a(s)$, $\cos \beta(s)$, then by definition

$$\frac{\partial G}{\partial n} = -\frac{\partial G}{\partial a} \cos \beta(s) + \frac{\partial G}{\partial b} \cos a(s),$$

$$u(x, y) + iv(x, y) = \frac{1}{2\pi} \int_0^{2\pi} u(s) \frac{\partial}{\partial n} \log \frac{c(z - c')}{z - c} \, ds + ic_2,$$

where c_2 is a real constant appearing because a conjugate function is only determined to within an additive constant. Similarly

$$v(x, y) - iu(x, y) = \frac{1}{2\pi} \int_0^{2\pi} v(s) \frac{\partial}{\partial n} \log \frac{c(z - c')}{z - c} \, ds - ic_1,$$

and multiplying this equation by i and adding it to the preceding, we have

$$2[u(x, y) + iv(x, y)] = 2f(z)$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(c) \frac{\partial}{\partial n} \log \frac{c(z - c')}{z - c} \, ds + c_1 + ic_2.$$

But

$$\frac{\partial}{\partial n} \log \frac{c(z - c')}{z - c} = - \frac{\partial}{\partial \rho} \log \rho e^{is} \frac{(z - e^{is}/\rho)}{z - \rho e^{is}}$$

and in this, after differentiating, we are to set $\rho = 1$.   This gives

$$- 1 - \frac{2e^{is}}{z - e^{is}}.$$

We have therefore

$$2f(z) = - \frac{2}{2\pi} \int_0^{2\pi} f(c) \frac{e^{is}}{z - e^{is}} \, ds - \frac{1}{2\pi} \int_0^{2\pi} f(c) ds + c_1 + ic_2.$$

To determine $c_1 + ic_2$, put $z = 0$, and note that according to potential theory

$$f(0) = \frac{1}{2\pi} \int_0^{2\pi} f(c) ds.$$

It appears that $c_1 + ic_2$ has this same value, and that the last three terms of the equation fall out, leaving

$$2f(z) = - \frac{2}{2\pi} \int_0^{2\pi} f(c) \frac{e^{is}}{z - e^{is}} \, ds,$$

$$\frac{\partial H}{\partial n} = - \frac{\partial H}{\partial a} \cos \beta(s) + \frac{\partial H}{\partial b} \cos a(s).$$

Then $\partial G/\partial n$ and $\partial H/\partial n$ satisfy equations (a) provided the order of differentiation of G and H with respect to a or b and with respect to x or y can be inverted.   This is clearly the case for the function in our problem so long as (x, y) keeps *within* the circle.

or, since $e^{is}ds = dc/i$ and $e^{is} = c$ on the circumference,

$$f(z) = \frac{1}{2\pi i}\int_0^{2\pi} f(c)\,\frac{dc}{c - z}\,.$$

The transition from the case of the circle to any region which can be conformally represented on the circle is easy, as Green's function is transformed into Green's function for the new region and $dc/(c - z)$ differs from the corresponding expression for the new region only by a function which disappears upon integration.

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BAUER'S ALGEBRA.

Vorlesungen über Algebra. Von GUSTAV BAUER. Herausgegeben vom Mathematischen Verein München. Leipzig, B. G. Teubner, 1903. vi + 376 pp.

THIS volume was planned in honor of the 80th birthday of Professor Bauer by the Mathematischer Verein of the students of the university and the technical high school of Munich. It presents in fact, not merely in title, lectures as actually given to students in their first or second year at the university, the course extending over two semesters. The preface is by Karl Doehlemann, who saw the book through the press at the request of the Verein.

Treating a wide range of subjects in a strictly elementary manner with many illustrative examples considered in detail, these lectures are certainly very attractive. If one can overlook the lack of rigor in two or three fundamental matters (discussed in detail below), one must regard the volume as one to be specially commended to beginners.

Part I (105 pages), is entitled "General properties of algebraic equations." The usual elementary theorems on complex quantities are given in 12 pages. In the construction of z^n by a series of similar triangles (page 13), n is restricted to positive integers, whereas the series may be continued in the opposite direction to give the negative integral powers. The mere statement that an elementary geometric construction for $z^{1/n}$ is impossible in general would attract the student more if accom-