ON THE CONDITION THAT A POINT TRANSFORMATION OF THE PLANE BE A PROJECTIVE TRANSFORMATION.

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1. WHILE it is well known that all projective transformations are collineations, the converse has, I believe, never been proved in all its generality. Möbius, in his "Barycentrischer Calcul" proves by means of his net of lines that if we start from any four independent * fixed points in the plane, we can either reach any point of the plane, by means of constructions with a ruler alone, or else come as near it as we please. Thence he infers that there cannot be more than one collineation which carries four given independent points into four other given positions. Since we know that there exists a projective transformation which carries over four arbitrarily given independent points into four other arbitrarily given independent points, we infer that every collineation is a projective transformation. In this reasoning, however, Möbius clearly assumes that he is dealing only with transformations which are in general one-to-one and continuous. There are, however, points in the two planes (the points on the vanishing lines) where the transformation is, strictly speaking, not defined. Thus two questions present themselves:

(1) Is it necessary to require that the collineation be continuous in order that Möbius's theorem be true?

(2) Throughout what part of the plane may we leave the transformation undefined?

It is my object in this paper to prove the following theorem, which answers the first question, and goes a long way toward answering the second.

Suppose we have a one-to-one correspondence between the points of two plane point sets S and S', each of which has an interior point, such that any three collinear points in either set have collinear images. Then the transformation of S into S' is a projective transformation.

* By four independent points in a plane we understand four points no three of which lie on a straight line.

2. It should be clearly understood that the points referred to in this theorem are actual geometric points, neither imaginary points nor points at infinity being considered. There is, however, no objection to our using the term "points at infinity" in the course of the proof to avoid long circumlocutions. In fact we shall find it convenient for this purpose to use the term "projective plane," which plane we understand to contain not merely the actual geometric points, but also an infinite number of points at infinity regarded as lying on a line according to the ordinary conventions of projective geometry.

I shall denote the images of points, etc., by primes, thus P' is the image of P, etc. I shall call the first plane I, the second II.

3. Before passing to the proof of the general theorem, it will be convenient to prove two special cases of it. The first is this :—

CASE A. *Suppose we have a one-to-one correspondence of the points of two planes, such that to every finite point of one corresponds one and only one finite point of the other, and, moreover, the images of collinear points are collinear. Then the transformation is projective.*

Before we can speak of a line L having a line L' as its image, we must make sure that not only do all the points on L correspond to points on a certain line L', but that they correspond to *all* of these points. This, however, is obvious when we recall that all the points on L' have images on L.

In the first place if two lines meet in I their images must meet in II, since their point of intersection has a finite image, and accordingly *the images of parallel lines are parallel.*

If we have three collinear points A, B, and C, such that $AB = BC$, then $A'B' = B'C'$. For if we draw two pairs of parallels through A and C we have a parallelogram, and the point B is determined by the intersection of its diagonals. In II we have a parallelogram with A' and C' for two opposite vertices and B' for the intersection of its diagonals. Therefore $A'B' = B'C'$, and *the image of a point midway between two points is midway between their images.*

The necessary and sufficient condition that six points lie on a conic is that the intersections of opposite sides of the hexagon formed by joining them should lie on a line. Since this condition involves nothing but collinearity of points, if it is true of one set of six points it is true of their images. Take any conic in one plane, select any five points on it, and consider the conic through their images. If we take a sixth point and let it describe the first conic, its image must always lie on the second conic. Therefore every point on the first conic has an image on the second, and, similarly every point on the second has an image on the first. Therefore one conic is the image of the other, and *the image of a conic is a conic.*

Since the center of a conic bisects every chord through it, its image possesses the same property and is, therefore, the center of the transformed conic, and *the image of a central conic is a central conic.* Since every line through the center of an ellipse cuts the ellipse, but this is not the case for the hyperbola, *an ellipse goes into an ellipse.*

If we have two points A and B on opposite sides of a straight line L, A' and B' lie on opposite sides of L'. For if they did not we could pass an ellipse through A' and B' which did not cut L'. Its image would be an ellipse through A and B not cutting L, which is impossible. *If, therefore, a point B lies between two other points A and C, B' lies between A' and C',* since A and C lie on opposite sides of any line through B.

If we have a set of collinear points equally spaced, their images are equally spaced and follow each other in the same order. For if L, M, N are any three points, L', M', N' are in the same order, for if L and N are on opposite sides of M, L' and N' are on opposite sides of M'. If A, B, C are three consecutive points, since $AB = BC$, $A'B' = B'C'$.

Now take any line in I and an origin O on it. Let P and A be any two points on the line, such that OP and OA are commensurable. Take the common measure of OP and OA and lay it off along OA. Suppose it is contained in OA n times and in OP m times. By laying off this common measure, we obtain a set of points equally spaced on OA. Their images must be a set equally spaced on $O'A'$. There must be n of these spaces in $O'A'$ and m in $O'P'$. Accordingly $O'P' : O'A' = m : n = OP : OA$.

Now suppose that OP and OA are incommensurable. Using as a measure of OP a sub-multiple of OA, which we ultimately allow to approach zero as its limit, we can use the familiar method to show that $O'P' : O'A' = OP : OA$, for we know that if P lies between the points reached by applying this measure k times and $k + 1$ times respectively, P' will lie between their images. *Hence distances along a fixed line from a given point are altered in a fixed ratio.*

Now suppose in the first plane we have a pair of rectangular axes with origin O, and a point P whose coördinates are (x, y). Consider their images in the second plane. We shall have a pair of axes, in general oblique, and a point P' referred to them. Denoting by A and B the orthogonal projection of P on OX and OY respectively, we have $OA = x$, $OB = y$. The image of the rectangle $OAPB$ is a parallelogram $O'A'P'B'$, and the coördinates $(\bar{x}, \bar{y})$ of P' are $\bar{x} = O'A'$, $\bar{y} = O'B'$. Then if l and m are the ratios by which distances are altered along OA and OB respectively, $\bar{x} = lx$, $\bar{y} = my$. If P' be referred to any pair of rectangular axes in the second plane, we have, by a transformation of coördinates,

$$x' = a_1x + b_1y + c_1,\ y' = a_2x + b_2y + c_2.$$

And this is a projective transformation. Thus the theorem A is proved.

4. Another special case is the following:

CASE B. *Suppose we have a one-to-one correspondence of all the points of two projective planes, such that to any three collinear points of one, correspond three collinear points of the other. Then the transformation is a projective transformation.*

If every point of I has a finite image, we have practically Case A. For by that case we see that the transformation of the finite points is effected by a projective transformation; and, since the points at infinity may be determined as the vertices of pencils of parallel lines, it is clear that this same projective transformation carries over the points at infinity to their new positions.

Now suppose that the image of the line at infinity in I is a line, L, in II. Apply to II any projective transformation that will throw L to infinity in a third plane III. To every point of I corresponds one and only one point of III. To any point at infinity in I corresponds a point in L in II, and, therefore, a point at infinity in III, and vice versa. Moreover any three collinear points in I have collinear images in II and hence in III, and vice versa. Then we have just such a transformation of I on III as we considered in the first case under Case B. Therefore the transformation of I on III is a projective transformation. Hence, since the inverse of a projective transformation is a projective transformation, and the succession of two projective transformations is a projective transformation, it follows that the transformation of I on II is projective. Thus theorem B is established.

5. Before proving the general theorem, I shall present a couple of lemmas.

Suppose we have three lines, α_1, α_2, α_3 (Fig. 1). Draw any other two lines, β_1, β_2, cutting α_1, α_2, α_3 in the points A and B, Q_1 and Q_2, E and F respectively. Let P_1 be the intersection of BQ_1

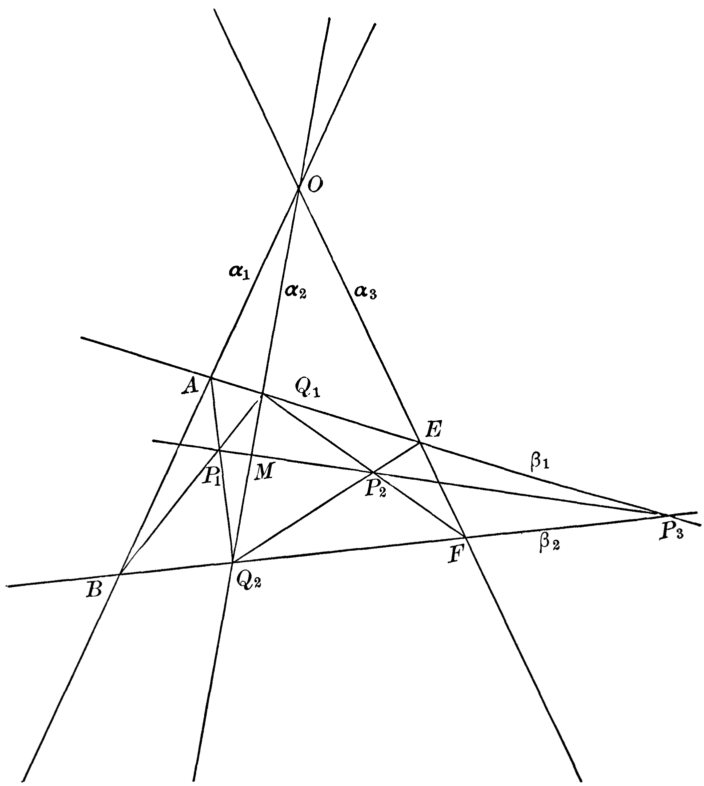

FIG. 1.

and AQ_2, P_2 of EQ_2 and FQ_1, P_3 of β_1 and β_2. *Then a necessary and sufficient condition that* α_1, α_2, α_3 *be concurrent is that* P_1, P_2, P_3 *be collinear.* This can be proved very easily by a projection, or is at once evident when we observe that P_1P_3 is the polar of the intersection of α_1, α_2, and P_2P_3 that of the intersection of α_2, α_3 with regard to the degenerate conic consisting of β_1 and β_2.

It is easy to obtain from this a condition for the collinearity of three points. If the points are P_1, P_2, P_3, choose any two points Q_1, Q_2. Let A be the intersection of Q_1P_3 and Q_2P_1, B that of Q_2P_3 and Q_1P_1, E that of Q_1P_3 and Q_2P_2, and F that of Q_2P_3 and Q_1P_2. We have just the same figure as we had before for the three lines, and since the collinearity of P_1, P_2, P_3 was both necessary and sufficient for the concurrence of α_1, α_2, α_3, the converse is true. Therefore *the necessary and sufficient condition for the collinearity of P_1, P_2, P_3 is the concurrence of AB, Q_1Q_2, EF.* It should be noticed that since β_2, P_3P_1, β_1, and the line joining the intersection of $\alpha_1\alpha_2$ with P_3 form an harmonic pencil, if Q_1 and Q_2 are on the same side of P_1P_3, the intersection of α_1, α_2, α_3 will lie between Q_1 and Q_2.

6. We are now in a position to prove the general theorem stated at the beginning. Suppose A is the interior point of the first plane; then we can describe a circle with A as center, which contains only points of the point set S. Call this circle C; all points of C have images.

My purpose is first to extend the definition of the transformation to all points of the projective plane, and thus to prove that it is a projective transformation by Case B.

Take any point P in the projective plane outside of C. We may determine it as the point of intersection of two lines cutting C. The points in C of either line have collinear images in II. Let us define the point P', the point of intersection of the two lines on which these images lie, as the image of P. To prove this definition permissible we must show that any two lines through P cutting C lead to the same point P', or, what is evidently equivalent, that any three lines through P, which cut C, determine in II three lines through P'. Call our three lines α_1, α_2, α_3, and apply our first test. Draw β_1 and β_2 so that they meet each other and also α_1, α_2, α_3 inside of C. Then evidently the points P_1 and P_2 lie within C, for A, B, Q_1, Q_2, E, F, are vertices of non-reëntrant quadrilaterals, which must have the points of intersection of their diagonals within C. Then since the points P_1, P_2, P_3 are in C and collinear, their images in II are collinear, and the three lines determined by the images of points on α_1, α_2, α_3 meet in a point P'.

If P is a point of S, it is evident that the image thus defined is the same as the image originally given. For if we take any two points in C collinear with it, their images are collinear with its image.

Now consider any three collinear points P_1, P_2, P_3; we wish to prove that their images, as we have just defined them, are collinear. If the line $P_1P_2P_3$ cuts C, these points are deter-

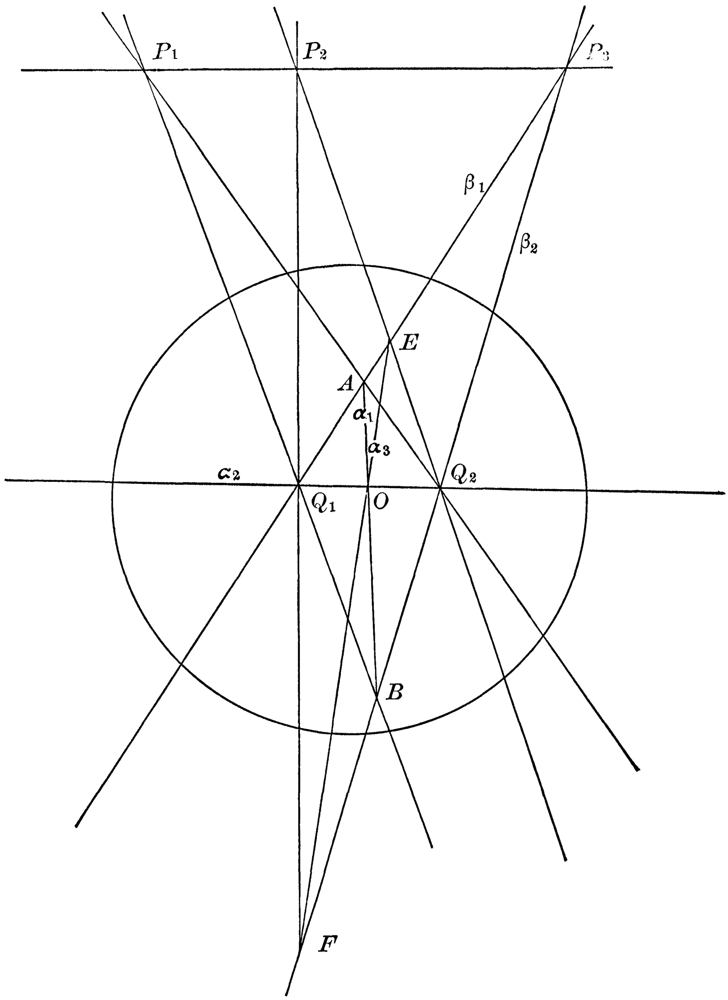

FIG. 2.

mined by the intersection of the line $P_1P_2P_3$ and any other lines through them cutting C. Consequently their images must lie on the line determined in II by the images of points on $P_1P_2P_3$.

If $P_1P_2P_3$ does not cut C, choose any two points Q_1 and Q_2 in C and apply our second test. The lines joining the intersections of Q_1P_3 and Q_2P_1 and Q_2P_3 and Q_1P_1, also of Q_1P_3 and Q_2P_2 and Q_2P_3 and Q_1P_2 must meet in a point on Q_1Q_2. This point must lie between Q_1 and Q_2 since $P_1P_2P_3$ does not cut Q_1Q_2 between Q_1 and Q_2, and accordingly lies in C. Since Q_1P_i, Q_2P_i pass through P_i their images must pass through P'_i. Since P_1Q_2, Q_1P_3, OA cut C and meet in a point, their images must meet in a point. Similarly $P'_1Q'_1$, $P'_3Q'_2$, $O'B'$ meet in a point. Similarly $E'F'$ passes through O'. Then evidently the condition for the collinearity of $P'_1P'_2P'_3$ is satisfied, and the *images of collinear points are collinear.*

Now since S' has an interior point we can apply all our reasoning to II and define an image of every point of it in I, and show that these images are collinear when the original points are. Have we thus established a one-to-one transformation of I and II? That is, if P is any point of I and P' its image in II, and if in turn P'' is the image in I of P', will P'' necessarily coincide with P? In the first place there is a one-to-one correspondence, by hypothesis, between S and S'. Take any point P not in S. We define P' by means of the images of sets of two points in S, each set collinear with P. Any two points collinear with P' must have images collinear with P''. Take the four points by which we defined P'. Their images in I determine P'' uniquely. But since they are points of S', they have as images the same points we started with, and P'' must coincide with P. Accordingly we have a one-to-one correspondence between the points of the projective planes I and II such that the images of three collinear points are collinear. Then by theorem B it follows that the transformation is projective, and thus the general theorem stated near the beginning of this paper is proved.

In conclusion I wish to express my gratitude to Professor Bôcher for aiding me in the preparation of this paper, by valuable suggestions and criticisms.

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