SERIES WHOSE PRODUCT IS ABSOLUTELY CONVERGENT.

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§ 1. That two absolutely convergent series yield an absolutely convergent product was first shown by Cauchy.* About three quarters of a century later Alfred Pringsheim pointed out that an absolutely convergent product may result also from the multiplication of a conditionally convergent or even a divergent series by an absolutely convergent series.† That the product of two conditionally convergent series, or of a conditionally convergent series and a divergent series, or of two divergent series, may be absolutely convergent was first made public by the present writer.‡ Thereupon Alfred Pringsheim treated the subject from a more general point of view and, by very simple methods, showed that the property in question is typical of certain classes of series.§ The present writer developed a new class of series possessing this property, demonstrated the validity of the fundamental laws of algebra in the multiplication of infinite series, and generalized a theorem of Abel on the multiplication of series.|| In the present article we aim to generalize some of the results previously obtained relating to absolutely convergent products of two or more series.

§2. In this investigation we shall start with an absolutely convergent series and determine pairs of series which are factors of the assumed series. Given the absolutely convergent series

$$U \equiv \sum_{n=0}^{n=\infty} u_n, \tag{1}$$

in which

$$u_n \equiv \sum_{r=0}^{r=n} c_r e_r c'_{n-r} e_{n-r},$$

* Analyse Algébrique, 1821, page 147.

† *Math. Annalen*, vol. 21 (1883), pp 357–359.

‡ *Transactions of the American Mathematical Society*, vol. 2, pp 25–36, January 1901; *Science*, new series, vol. 14, p. 395 September 13, 1901.

§ *Transactions of the American Mathematical Society*, vol. 2, pp. 404–412, October, 1901.

|| BULLETIN, 2d series, vol. 8, pages 231–236, March, 1902.

e_r is the general term of any absolutely convergent series, say

$$e_r \equiv \frac{1}{(z+r)[\log(z+r)]^\lambda},$$

$\lambda > 1$, and c_r, c'_r are constants either real or complex. That series (1) is absolutely convergent becomes evident, if we observe that it is the product formed according to Cauchy's multiplication rule, of the two absolutely convergent series

$$\sum_{r=0}^{r=\infty} c_r e_r \quad \text{and} \quad \sum_{r=0}^{r=\infty} c'_r e_r.$$

Let us assume

$$a_{r-s} - a_r = c_r e_r, \tag{2}$$

$$h_r(b_{r-t} - b_r) = c'_r e_r,$$

where s and t are positive integers, a and b are real or complex numbers and h_r is an odd or an even power of -1. Let it be agreed that a and b cannot have negative subscripts; in other words, that $a_{-x} = b_{-x} = 0$. We have

$$u_n \equiv \sum_{r=0}^{r=n} c_r e_r \cdot c'_{n-r} e_{n-r} = \sum_{r=0}^{r=n} h_{n-r}(a_{r-s} - a_r)(b_{n-r-t} - b_{n-r}).$$

If we perform the indicated multiplications and collect the coefficients of a_r, we obtain

$$u_n \equiv \sum_{r=0}^{r=n} a_r(-h_{n-r}b_{n-r-t} + h_{n-r}b_{n-r} + h_{n-r-s}b_{n-r-s-t} - h_{n-r-s}b_{n-r-s}). \tag{3}$$

If we assume $h_x = -h_{x+t}$, we have

$$u_n \equiv \sum_{r=0}^{r=n} (a_{r-t} + a_r)(h_{n-r}b_{n-r} - h_{n-r-s}b_{n-r-s}). \tag{4}$$

It will be noticed that if, in the two terms in (3) which involve the factors a_r and a_{r-t}, respectively, we remove the parentheses, we obtain eight terms which are distributed among three terms of the series in (4), namely the three terms which involve, respectively, the parentheses

$$(a_{r-2t} + a_{r-t}), \quad (a_{r-t} + a_r), \quad (a_r + a_{r+t}).$$

From the inspection of series (4) we readily see that the series (1) may be considered to be the product of the following two series :

$$\sum_{r=0}^{r=\infty} (a_{r-t} + a_r) \equiv a_0 + a_1 + \cdots + a_{t-1} + (a_0 + a_t) \\ + (a_1 + a_{t+1}) + \cdots \tag{5}$$

and

$$\sum_{r=0}^{r=\infty} (h_r b_r - h_{r-s} b_{r-s}) \equiv h_0 b_0 + h_1 b_1 + \cdots + h_{s-1} b_{s-1} \\ + (h_s b_s - h_0 b_0) + \cdots. \tag{6}$$

According to the condition $h_r = - h_{r+t}$, we are permitted to choose any sign we please for any t consecutive factors h_r. After such a choice has been made, the signs represented by any of the other factors h_x are determined.

Since relations (2) are the only conditions imposed upon the values of a and b, it is possible to choose these values so that each of the series (5) and (6) is absolutely convergent, conditionally convergent, or divergent. Thus, if $|a_r|$ is of the order of magnitude e_r, series (5) is absolutely convergent; if

$$|a_{r-t} + a_r| \geqq \frac{1}{(r+2)\log(r+2)},$$

but approaches the limit zero as r increases indefinitely and if $(a_{r-t} + a_r)$ is opposite in sign to and has greater numerical value than $(a_{r-t+1} + a_{r+1})$, then (5) is conditionally convergent; if $|a_{r-t} + a_r|$ does not approach the limit zero for all values of r, as r increases indefinitely, the series (5) is divergent. Similarly for (6). Yet in every case, the product of (5) and (6) is absolutely convergent.

Since, so far as we know, no two divergent series with *complex* terms have before been given, whose product is absolutely convergent, it may be well to construct a special example. Let

$a_r = b_r = A + B_r e_r$, where A and B_r are complex constants. Moreover, let $s = 1$ and $t = 2$, $h_0 = -h_1 = +1$. It will be seen that values which are not infinite can be assigned to the coefficients c_r and c'_r so that equations (2) are satisfied. By substitution in the two series (5) and (6) we obtain the following two series :

$$\left.\begin{aligned} S_1 \equiv (A + B_0 e_0) + (A + B_1 e_1) + (2A + B_2 e_2 + B_0 e_0) \\ + (2A + B_3 e_3 + B_1 e_1) + \cdots, \\ S_2 \equiv (A + B_0 e_0) - (2A + B_1 e_1 + B_0 e_0) - (B_2 e_2 - B_1 e_1) \\ + (2A + B_3 e_3 + B_2 e_2) + (B_4 e_4 - B_3 e_3) - \cdots. \end{aligned}\right\} \quad (7)$$

It will be seen that both series in (7) are divergent and complex, and that their product is absolutely convergent. Another pair of complex series possessing this property is given at the close of this article.

Since all the terms in the series S_1 are preceded by the positive sign, it is readily seen that any positive integral power of S_1 is a divergent series whose terms increase numerically without limit as r increases without limit. The same conclusion holds for the series S_2. Since $S_1 \cdots S_2 \cdots S_1 \cdots S_2 \cdots$ [to p pairs of factors] $= (S_1 S_2)(S_1 S_2) \cdots$ [to p parentheses] $= S_1^p \cdots S_2^p$, and since $S_1 \cdots S_2$ is an absolutely convergent complex product, it follows that the product of the two complex series S_1^p and S_2^p, the terms of both of which increase numerically without limit as r increases without limit, is absolutely convergent, no matter how large a value the integer p may have.*

If we let a_r and b_r be positive and decreasing monotonously toward zero in such a way that Σa_r and Σb_r are both divergent, if moreover, $t = 2$, $s = 1$, $h_0 = +1$, $h_1 = -1$, then (5) and (6) reduce to the two following series, one divergent, the other conditionally convergent, given by Pringsheim :†

$$\sum_{r=0}^{r=\infty} (a_{r-2} + a_r) \text{ and } \sum_{r=0}^{r=\infty} (-1)^{r+\left[\frac{r-1}{2}\right]} (b_{r-1} + (-1)^{r-1} b_r).$$

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* See BULLETIN, 2d series, vol. 8 (1902), pp. 233–236.

† See *Transactions of the American Mathematical Society*, vol. 2, p. 408, equation (B). The notation $\left[\frac{r-1}{2}\right]$ signifies here the largest integer contained in $\frac{r-1}{2}$.

In order to deduce from (5) and (6) the pair of conditionally convergent series, whose product is absolutely convergent, which we gave in the BULLETIN, volume 8 (1902), page 231, let $s = t = 4$; $h_0 = h_1 = h_2 = h_3 = +1$; $r = 4v, 4v+1, 4v+2$, or $4v+3$; $r' = 4v$. Let moreover the parenthesis $(a_{r-t} + a_r)$, which we represent for convenience by a'_r, be a real number which is positive when $r = 4v$ or $4v+1$ and negative when $r = 4v+2$ or $4v+3$, and such that $|a'_r| \leqq r^{-u}$, where $\frac{1}{2} < u \leqq 1$, and $\Sigma |a'_r|$ is divergent. Without violating conditions (2) we may assume further

$$a'_{r'} = a'_{r'+1}, \quad |a'_{r'+2}| - a'_{r'} = e_{r'} \pm w_{r'},$$

$$a'_{r'+2} = a'_{r'+3}, \quad |a'_{r'+2}| - a'_{r'+4} = e_{r'+4} \pm w_{r'+4},$$

where w_r is a quantity numerically not greater than

$$\frac{1}{(2+r)^{2-u}[\log(2+r)]^{\lambda}}.$$

Similarly we may assume, without violating conditions (2),

$$h_r b_r - h_{r-s} b_{r-s} \equiv b'_r, \quad |b'_r| \leqq r^{-u}, \quad \sum_{r=0}^{r=\infty} |b'_r| \text{ divergent},$$

b'_r a real number which is positive when $r = 4v$ and $4v+2$, and negative when $r = 4v+1$ and $4v+3$; also

$$b'_{r'} = b'_{r'+2}, \quad b'_{r'} - |b'_{r'+1}| = e_{r'} \pm w_{r'},$$

$$b'_{r'+1} = b'_{r'+3}, \quad b'_{r'+4} - |b'_{r'+1}| = e_{r'+4} \pm w_{r'+4}.$$

There result from these assumptions the two conditionally convergent series $\sum_{r=0}^{r=\infty} a'_r$ and $\sum_{r=0}^{r=\infty} b'_r$, whose product is an absolutely convergent series, which were previously given by us in the last mentioned article.

§7. If we are given a series

$$\sum_{r=0}^{r=\infty} (a_{r-t} + a_r),$$

such that $a_{r-s} - a_r = c_r c_r$, it is quite evident that we can always find a mate for it, such that the product of the two shall be absolutely convergent.

Again it is possible to find two series, every term in the product of which, after the $(t + 1)$th term, vanishes identically. Moreover, the two series may be so chosen that the product possesses the additional property of having for its sum any desired finite number N. To bring this about modify conditions (2) thus,

$$a_{r-s} - a_r = cd^r,$$

where d is any number, and, for $r \geqq t$,

$$h_r(b_{r-t} - b_r) = 0.$$

Then $\sum_{r=0}^{r=\infty} u_r$ is still absolutely convergent. Let t be even and

$$b_{t-1} = db_{t-2} = d^2 b_{t-3} = \cdots = d^{t-1} b_0,$$
$$h_0 = -h_1 = h_2 = -h_3 = \cdots = -h_t = 1.$$

Then we have, for $r \geqq t$,

$$u_r = b_0 cd^r - b_0 d \cdot cd^{r-1} + b_0 d^2 \cdot cd^{r-2} - \cdots - b_0 d^{t-1} \cdot cd^{r-t+1} \equiv 0.$$

Under the above conditions the two series (5) and (6) have a product, every term of which, after the $(t + 1)$th term, vanishes identically.

To make the sum of the product of the two series (5) and (6) equal to N, place the sum of the first $t + 1$ terms in the product equal to N and then determine the values of d which satisfy this condition.

Thus, let $t = 2$, $s = 1$, $c = +1$, then from $a_{r-1} - a_r = d^2$, for $r \geqq 1$, we get the series (5) (the first term of which is assumed to be a_0)

$$a_0 + (a_0 - d) + (2a_0 - d - d^2) + (2a_0 - 2d - d^2 - d^3)$$
$$+ (2a_0 - 2d - 2d^2 - d^3 - d^4) + \cdots.$$

Assuming $b_0 = 1$, the series (6) becomes

$$1 - (d + 1) + (d - 1) + (d + 1) - (d - 1) - (d + 1)$$
$$+ (d - 1) + \cdots.$$

All the terms in the product of these two series, after the second term, vanish. Putting the sum of the first two terms $= N = 0$, we have $a_0 - d(a_0 + 1) = 0$. If we assume $a_0 = -\frac{10}{9}$, then $d = 10$, and the two factor series become

$$U = -\tfrac{10}{9} - 11\tfrac{1}{9} - 112\tfrac{2}{9} - \cdots$$
$$U' = 1 - 11 + 9 + 11 - 9 - 11 + 9 + 11 - \cdots$$

Since $U \cdot U' = 0$, we have $U^p \cdot U'^p = 0$. As all the terms in U are of the same sign, it is easily seen that U^p is divergent for all positive integral values of p. U' and U'^p are also divergent.

§ 8. If we assume $t = 2$, $s = 1$, $c = -1$, $a_0 = 1$, $b_0 = 1$, $N = 0$, then the condition that the sum of the product of (5) and (6) shall vanish becomes $d^2 + 1 = 0$ and (letting $i = \sqrt{-1}$) the factor series thus obtained are the two complex divergent series

$$1 + (1 + i) + i + 0 + 1 + (i + 1) + i + 0 + \cdots$$
$$1 - (1 + i) + (i - 1) + (i + 1) - (i - 1) - (i + 1) + \cdots$$

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THREE SETS OF GENERATIONAL RELATIONS DEFINING THE ABSTRACT SIMPLE GROUP OF ORDER 504.

BY PROFESSOR L. E. DICKSON.

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1. CONSIDERABLE interest attaches to the simple group of order 504. The existence of this simple group was discovered by Professor Cole.* This was one of the facts that lead Professor Moore † to his investigation of the linear fractional group in the general Galois field, resulting in the discovery of a new doubly infinite system of simple groups.

* "On a certain simple group," Mathematical Papers, Chicago Congress of 1893.

† BULLETIN, December, 1893; Mathematical Papers, Congress of 1893.