INFINITESIMAL DEFORMATION OF THE SKEW HELICOID.

BY DR. L. P. EISENHART.

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CONSIDER the skew helicoid S, defined by the equations

$$(1) \qquad x = u \cos v, \quad y = u \sin v, \quad z = av.$$

We shall show that the problem of the infinitesimal deformation of this surface can be completely solved.

By direct calculation we find

$$(2) \qquad E = \sum \left(\frac{\partial x}{\partial u}\right)^2 = 1, \quad F = \sum \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} = 0,$$
$$G = \sum \left(\frac{\partial x}{\partial v}\right)^2 = u^2 + a^2,$$

and

$$(3) \qquad X, Y, Z = \frac{a \sin v, \, -a \cos v, \, u}{\sqrt{u^2 + a^2}}$$

where Y, X, Z denote the direction cosines of the normal. Again we find

$$(4) \qquad D = \sum X \frac{\partial^2 x}{\partial u^2} = 0, \quad D' = \sum X \frac{\partial^2 x}{\partial u \, \partial v} = \frac{-a}{\sqrt{u^2 + a^2}},$$
$$D'' = \sum X \frac{\partial^2 x}{\partial v^2} = 0.$$

The characteristic equation of the deformation reduces in this case to

$$\frac{\partial^2 \phi}{\partial u \partial v} + \frac{u}{u^2 + a^2}\frac{\partial \phi}{\partial v} = 0,$$

of which the general integral is

$$(5) \qquad \phi = \frac{U + V}{\sqrt{u^2 + a^2}},$$

where U is a function of u alone and V is a function of v alone.

The cartesian cöordinates of the surface S_1, corresponding to S with orthogonality of linear elements, have the following expressions:*

$$x_1 = (U + V) \sin v - 2 \int \sin v \cdot V' dv,$$

$$(6) \qquad y_1 = -(U + V) \cos v + 2 \int \cos v \cdot V' dv,$$

$$z_1 = -\frac{1}{a} \left[(U + V) u - 2 \int u \cdot U' du \right],$$

where the accent denotes differentiation. From (6) we have that, when V is a constant, S_1 is a surface of revolution. Moreover, since these formulæ involve an arbitrary function of u, it follows that any surface of revolution can be defined by them.

Conversely, *given a surface of revolution defined by*

$$x = u \cos v, \quad y = u \sin v, \quad z = U;$$

the helicoid with plane director, whose equations are

$$\bar{x} = U_1 \sin v, \quad \bar{y} = -U_1 \cos v, \quad \bar{z} = av,$$

has the same axis and corresponds with orthogonality of linear elements, if

$$U_1 = au \int \frac{U'}{u^2} du,$$

where the accent denotes differentiation with respect to u.

By direct calculation we find from (6),

$$(7) \quad F_1 = \sum \frac{\partial x_1}{\partial u} \frac{\partial x_1}{\partial v} = \frac{V'}{a^2} \left[u (U + V) - (u^2 + a^2) U' \right].$$

From (4) we see that the lines $u =$ const., $v =$ const. on S are asymptotic, and consequently the corresponding lines on S_1 form a conjugate system. Hence it follows from (7) that *the necessary and sufficient condition that asymptotic lines on S cor-*

* Bianchi, Lezioni, p. 276.

respond to lines of curvature on S_1 is that $V' = 0$, that is, S_1 must be a surface of revolution.

From (5) we see that in the latter case ϕ is a function of u alone. This, however, is a general property of the infinitesimal deformation of minimal surfaces. For, from the following formula, which we have established elsewhere,*

$$F_1 = F\phi^2 + \frac{1}{K^2(EG - F^2)}\left(D\frac{\partial\phi}{\partial v} - D'\frac{\partial\phi}{\partial u}\right)\left(D'\frac{\partial\phi}{\partial v} - D''\frac{\partial\phi}{\partial u}\right),$$

it is seen that when S is a minimal surface referred to its asymptotic lines, the necessary and sufficient condition that the parametric lines on S_1 be the lines of curvature is that ϕ shall be a function of u alone or a function of v alone.

When in particular we take

$$(8) \qquad U = \sqrt{u^2 + a^2}, \quad V = 0,$$

we get

$$x_1 = \sqrt{u^2 + a^2} \cdot \sin v, \quad y_1 = -\sqrt{u^2 + a^2} \cdot \cos v,$$
$$z_1 = -a \log(u + \sqrt{u^2 + a^2}),$$

which define the catenoid. From (5) we get $\phi = 1$, which is the case whenever, in the deformation of a minimal surface, the adjoint of the latter is taken for the surface S_1.†

Genty ‡ has shown that the cartesian coördinates, x_0, y_0, z_0, of the associate surface § S_0 in an infinitesimal deformation are given by the equations

$$dx_1 = z_0 dy - y_0 dz,$$
$$dy_1 = x_0 dz - z_0 dx,$$
$$dz_1 = y_0 dx - x_0 dy.$$

Substituting the expressions for $x, y, \cdots, z_1$, from (1) and (6), and solving we find

$$(9) \qquad \begin{aligned} x_0 &= \frac{1}{a}\left[(U + V - uU')\sin v + V' \cdot \cos v\right], \\ y_0 &= \frac{1}{a}\left[-(U + V - uU')\cos v + V' \cdot \sin v\right], \\ z_0 &= U'. \end{aligned}$$

* *Amer. Jour. of Math.*, vol. 24, p. 177.
† *Ibid.*, p. 192.
‡ Toulouse *Annales*, vol. 9.
§ Bianchi, *l. c.*, p. 279.

The linear element of S_0 is readily found to be

$$(10)\quad ds_0^{\,2} = \frac{U''^2}{a^2}(u^2 + a^2)\,du^2 + \frac{1}{a^2}(V'' + V - uU' + U)^2\,dv^2.$$

It is well known that the lines upon any associate surface corresponding to the asymptotic lines on S form a conjugate system. From (10) we see that the conjugate system on S_0 corresponding to asymptotic lines on S are the lines of curvature. Furthermore, the lines of curvature $v =$ const. are geodesics and consequently S_0 is a surface of Monge.* From the form of the coefficient of dv^2 in (10) we have that the generating developable is a cylinder.† Hence

In any infinitesimal deformation of a skew helicoid the associate surface is a moulure surface.

Conversely, given any moulure surface; its equations can be put in the form (9) and then can be taken for the associate surface in the deformation of the helicoid (1), corresponding to the value $(U + V)(u^2 + a^2)^{-\frac{1}{2}}$ of the characteristic function.

From (6) and (9) we get the

THEOREM: *When the surface S_1 in an infinitesimal deformation of a skew helicoid is a surface of revolution, the associate surface S_0 also is a surface of revolution, and their lines of curvature correspond.*

And conversely,

When the associate surface S_0 is a surface of revolution, the characteristic surface S_1 is a surface of revolution.

When in particular S_1 is the catenoid, S_0 is the sphere of radius unity and center at the origin.

If we put

$$U = a\sqrt{u^2 + a^2} - au\log(u + \sqrt{u^2 + a^2}), \qquad V = 0,$$

the formulæ (9) define the catenoid. We have shown‡ that the necessary and sufficient condition that the lines of curvature be unaltered in the deformation of S is that S_0 be the adjoint minimal surface of S. Hence, when

$$\phi = a\left[1 - \frac{u}{\sqrt{u^2 + a^2}}\log(u + \sqrt{u^2 + a^2})\right],$$

* Monge, Applic. de l'Analyse a la Géométrie 5 ed., chap. 25.
† Darboux, Leçons, vol. 1, p. 105.
‡ L. c., p. 199.

the corresponding deformation of S leaves the lines of curvature unaltered and only in this case.

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ON INTEGRABILITY BY QUADRATURES.

BY DR. SAUL EPSTEEN.

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THE object of this note is to show that Vessiot's noted theorem that: "the necessary and sufficient condition that a linear differential equation shall be integrable by quadratures is that its group of rationality shall be integrable," * is a special case of the Jordan-Beke † theorem on reducibility of differential equations.

The Jordan-Beke theorem is to the effect that "if a linear differential equation is reducible in the sense of Frobenius ‡ then its group of rationality will transform a certain linear manifoldness of the solutions (which does not include the total n-dimensional manifoldness) into itself."

Analytically interpreted § this says that the group

$$
(1)\quad
\begin{aligned}
y_1 &= a_{11}y_1 + \cdots + a_{1k}y_k, \\
&\cdots\cdots\cdots \\
y_k &= a_{k1}y_1 + \cdots + a_{kk}y_k, \\
y_{k+1} &= a_{k+1,1}y_1 + \cdots + a_{k+1,k}y_k + a_{k+1,\,k+1}y_{k+1} + \cdots + a_{k+1,n}y_n, \\
&\cdots\cdots\cdots \\
y_n &= a_{n1}y_1 + \cdots + a_{nk}y_k + a_{n,\,k+1}y_{k+1} + \cdots + a_{nn}y_n,
\end{aligned}
$$

is isomorphic with the group of rationality. For convenience it is well to adopt Loewy's notation, writing for (1) simply the coefficients

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* Vessiot: *Ann. de l'Ec. nor. sup.*, 1892.

† C. Jordan. *Bull. de la Soc. Math. de France*, vol. 2; Beke: *Math. Annalen*, vol. 45, p. 279.

‡ Frobenius: *Crelle*, vol. 76.

§ A. Loewy: "Ueber die irreduciblen Factoren," etc., *Berichte der math.-phy. Classe der Königl. Sächs. Gesellschaft der Wissenschaften zu Leipzig*, vol. 54 (1902), pp. 1–13.