the transcendentals corresponding to π and e in the nth process are algebraic numbers borrowed from the $(n-1)$th and $(n+1)$th processes respectively. The totality of numbers may thus be conceived as arranged in an infinite number of strata, the ordinary algebraic numbers forming a single stratum.

With the extension of the differentiation process, and the stratification of the number body, all the results of the ordinary calculus, for example Taylor's theorem, can be expressed in terms of any two consecutive processes.

<div align="right">THOMAS F. HOLGATE,
Secretary of the Section.</div>

EVANSTON, ILL.

THE VALUE OF $\int_0^{\pi/2} (\log 2 \cos \varphi)^m \varphi^n d\varphi$.

BY PROFESSOR F. MORLEY.

THE integral in question, in which m is any positive integer, and n is any even positive integer or zero, is given in effect, for the cases $m = 1$, $n = 0$, in Williamson's Integral Calculus, §118 of the second edition, being taken from a paper by H. G. (presumably Harvey Goodwin, Bishop of Carlisle) in volume 3 of the *Quarterly Journal of Mathematics*. Further it is given in effect, for the case $m = 2$, $n = 0$, in Wolstenholme's Problems, p. 332 of the second edition. It seems worth while to show how it can be expressed in general, in terms of the constants $s_r = \sum_1^\infty 1/n^r$, which may be regarded as known.

We know that when p and q are real

$$1 + \frac{p \cdot q}{1 \cdot 1} + \frac{p(p-1)q(q-1)}{1 \cdot 2 \cdot 1 \cdot 2} + \cdots = \frac{\Gamma(1 + p + q)}{\Gamma(1 + p)\Gamma(1 + q)},$$

when $p + q > -1$ (Forsyth, Differential equations, p. 197). But the left member is the constant term in the product of the series

$$(1 + x)^p = 1 + px + \cdots$$

and

$$(1 + 1/x)^q = 1 + q/x + \cdots.$$

We suppose p and q positive, so that both series are absolutely convergent when $|x| = 1$.   We may then multiply the series and get a Laurent series for

$$(1 + x)^p(1 + 1/x)^q.$$

Hence

$$\frac{1}{2i\pi} \int_{|x|=1} (1 + x)^p(1 + 1/x)^q \frac{dx}{x} = \frac{\Gamma(1 + p + q)}{\Gamma(1 + p)\Gamma(1 + q)}.$$

The path of integration shall be from -1 positively round the circle; the point -1 being a branch point of the integral, but not an infinity.

Writing $x = \exp 2i\varphi$, the integral becomes

$$\frac{1}{\pi} \int_{-\pi/2}^{\pi/2} (2 \cos \varphi)^{p+q} \exp i(p - q)\varphi d\varphi.$$

Let

$$p + q = 2\alpha, \quad p - q = 2\beta ;$$

then

$$\frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \exp\{2\alpha \log (2 \cos \varphi) + 2i\beta\varphi\}d\varphi$$
$$= \frac{\Gamma(1 + 2\alpha)}{\Gamma(1 + \alpha + \beta)\Gamma(1 + \alpha - \beta)}.$$

Both sides can be developed in the form

$$\Sigma\Sigma C_{m,\,n}\alpha^m\beta^n ;$$

the left directly by the exponential theorem, giving

$$C_{m,\,n} = \frac{2^{m+n}}{\pi m!\; n!} \int_{-\pi/2}^{\pi/2} (\log 2 \cos \varphi)^m (i\varphi)^n d\varphi,$$

while the right, obtained from the formula

$$\log \Gamma(1 + n) = -\gamma n + \sum_2^\infty (-)^r \frac{s_r}{r} x^r,$$

is

$$\exp\left[\sum_2^\infty (-)^r \frac{s_r}{r} \{(2\alpha)^r - (\alpha + \beta)^r - (\alpha - \beta)^r\}\right].$$

Hence the result: The integral

$$\frac{2}{\pi} \int_0^{\pi/2} (\log 2 \cos \varphi)^m \varphi^n d\varphi$$

where m is any positive integer, n any positive even integer or 0, is

$$(-)^{n/2} \frac{m! \; n!}{2^{m+n}} \times \text{ the coefficient of } a^m \beta^n \text{ in}$$

$$\exp \left[\sum_{2}^{\infty} (-)^r \frac{s_r}{r} \{ (2a)^r - (a + \beta)^r - (a - \beta)^r \} \right].$$

This generating function is, to a few terms,

$$\exp \left[s_2 (a^2 - \beta^2) - 2 \; a s_3 (a^2 - \beta^2) \right.$$
$$\left. + \tfrac{1}{2} s_4 \left(7 a^4 - 6 a^2 \beta^2 - \beta^4 \right) - 2 \; a s_5 (3 a^4 - 2 a^2 \beta^2 - \beta^4) + \right]$$

or

$$1 + s_2 (a^2 - \beta^2) - 2 s_3 (a^3 - a \beta^2)$$

$$+ \frac{s_2{}^2 + 7 s_4}{2} a^4 - (s_2{}^2 + 3 s_4) a^2 \beta^2 + \frac{s_2{}^2 - s_4}{2} \beta^4$$

$$- 2 (s_2 s_3 + 3 s_5) a^5 + 4 (s_2 s_3 + s_5) a^3 \beta^2 - 2 (s_2 s_3 - s_5) a \beta^4 + \cdots$$

Thus, in tabular form, a few values for

$$\frac{2}{\pi} \int_0^{\pi/2} (\log 2 \cos \varphi)^m \varphi^n d\varphi$$

are :

$n =$	0	2	4
$m = 0^*$	1	$\frac{1}{2} s_2$	$\frac{3}{4}(s_2{}^2 - s_4)$
1	0	$- \frac{1}{2} s_3$	$- \frac{3}{2}(s_2 s_3 - s_5)$
2	$\frac{1}{2} s_2$	$\frac{1}{4}(s_2{}^2 + 3 s_4)$	
3	$- \frac{3}{2} s_3$	$- \frac{3}{2}(s_2 s_3 + s_5)$	
4	$\frac{3}{4}(s_2{}^2 + 7 s_4)$		
5	$- \frac{15}{2}(s_2 s_3 + 3 s_5)$		

ON THE ALGEBRAIC POTENTIAL CURVES.

BY DR. EDWARD KASNER.

THE object of this paper is to derive the characteristic geometric properties of a class of curves which are of in-

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* The row for which $m = 0$ is of course merely a verification, leading to the known values
$$s_2 = \pi^2/6, \quad s_4 = \pi^4/90.$$