THE APRIL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

A REGULAR meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York City on Saturday, April 27, 1901. Thirty-four persons were present at the two sessions, including the following thirty-one members of the Society :

Mr. Joseph Allen, Dr. J. E. Clarke, Professor F. N. Cole, Dr. W. S. Dennett, Professor T. S. Fiske, Mr. A. S. Gale, Dr. G. B. Germann, Mr. F. A. Giffen, Dr. W. A. Granville, Miss Ida Griffiths, Professor James Harkness, Dr. Edward Kasner, Mr. C. J. Keyser, Professor Pomeroy Ladue, Dr. G. H. Ling, Dr. Emory McClintock, Dr. James Maclay, Dr. G. A. Miller, Professor Frank Morley, Professor W. F. Osgood, Miss Ida Schottenfels, Professor C. A. Scott, Dr. Virgil Snyder, Dr. H. F. Stecker, Professor J. H. Tanner, Professor H. D. Thompson, Professor E. B. Van Vleck, Professor J. M. Van Vleck, Professor L. A. Wait, Miss E. C. Williams, Professor R. S. Woodward.

Vice-President Thomas S. Fiske occupied the chair. The Council announced the election of the following persons to membership in the Society : Mr. C. W. McG. Black, Harvard University, Cambridge, Mass.; Dr. S. E. Slocum, University of Cincinnati, Cincinnati, Ohio. Two applications for membership were received.

To relieve the increasing burden of administration, the office of Assistant Secretary was created, and filled by the appointment of Dr. Edward Kasner, to serve until February, 1902.

The library of the Society, which at present consists mainly of some five hundred unbound volumes of journals received as exchanges, is about to be deposited in the library of Columbia University, under an agreement by which the University undertakes to bind, catalogue, and care for the books now on hand and all future additions, and to make them easily accessible to the members of the Society. Arrangements will be made by which the books may be temporarily loaned to members living at a distance. The library is to be kept as a separate collection, duplicating as far as may be the general University library, and aiming to become as complete as possible in itself. The title to the books remains in the Society, which reserves the right to withdraw them under agreed conditions.

The following papers were read at this meeting:

(1) Dr. W. A. GRANVILLE: "Invariants of some m-gons under certain projective Lie groups in the plane."

(2) Dr. EDWARD KASNER: "The algebraic potential surfaces."

(3) Professor F. MORLEY: "On the real foci of algebraic curves."

(4) Mr. GEORGE PEIRCE: "A curious approximate construction for π."

(5) Professor E. W. HYDE: "On a surface of the sixth order, which is touched by all screws belonging to a three-conditioned system."

(6) Professor L. E. DICKSON: "The hyper-orthogonal groups."

(7) Professor E. W. BROWN: "On least action and minimal surfaces."

(8) Professor W. H. METZLER: "On certain aggregates of determinant minors."

(9) Professor E. B. VAN VLECK: "On the convergence of continued fractions with complex elements; supplementary note."

(10) Professor W. F. OSGOOD: "On a fundamental property of a minimum in the calculus of variations and the proof of a theorem of Weierstrass."

(11) Professor E. O. LOVETT: "The geometry of quadrics."

(12) Professor E. O. LOVETT: "The differential geometry of n-dimensional space."

(13) Dr. G. A. MILLER: "On the groups generated by two operators."

(14) Dr. EDWARD KASNER: "The relations between the angles of any number of lines in n-space."

(15) Dr. L. P. EISENHART: "Isothermal conjugate systems of lines on surfaces."

(16) Dr. E. J. WILCZYNSKI: "Geometry of a simultaneous system of two linear homogeneous differential equations of the second order."

(17) Dr. H. F. BLICHFELDT: "A new determination of the primitive continuous groups in two variables."

Mr. Peirce's paper was communicated to the Society by Professor E. W. Brown. In the absence of the authors, the papers of Mr. Peirce, Professor Hyde, Professor Dickson, Professor Brown, Professor Metzler, Professor Lovett, Dr. Eisenhart, Dr. Wilczynski, and Dr. Blichfeldt were read by title. The papers of Mr. Peirce and Dr. Miller will appear in the BULLETIN. Abstracts of the other papers are given below.

Dr. Granville finds the invariant functions of certain configurations under the substitutions of some of the important subgroups of the general projective group in the plane. Problems of this sort are among the most important and interesting in Lie's theory of continuous groups. First an outline is given of the theory followed in solving the problems. Then the invariant functions of a quadrangle under the substitutions of the group

$$\left|\overline{xp,\ yp,\ xq,\ yq,\ x^2p + xyq,\ xyp + y^2q}\right|$$

are found. The results do not agree with those found by Professor Lovett in the *Annals of Mathematics*, volume 12, where the same problem is proposed. The remaining four problems solved are of the same nature.

The surfaces considered by Dr. Kasner in his first paper are obtained by equating the rational integral potential functions $\varphi(x, y, z)$ to zero; their analytic property is expressed by the equation

$$\Delta\varphi \equiv \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = 0.$$

It is shown that the surfaces may be defined geometrically by their relation to the imaginary circle at infinity: the surfaces are apolar to this circle when the latter is considered as a degenerate surface of the second class. The potential surfaces of the second order are rectangular hyperboloids, *i. e.*, their asymptotic cones contain triples of mutually orthogonal generators. The polar quadrics of any potential surface are rectangular hyperboloids; conversely, a surface is potential when all its polar quadrics are rectangular hyperboloids. Potential cylinders and cones are considered in detail. The results are finally extended to n-dimensional space.

The point of Professor Morley's note was that, given any line equation of an algebraic curve

$$f(\xi_1, \cdots, \xi_n) = 0,$$

where ξ_r is the distance from a fixed point a_r to the moving line, the real foci are given by writing in the terms of highest order $x - a_r$ for ξ_r, where $x - a_r$ is the stroke from a fixed point to a focus. Some applications of this simple rule were given.

Any three screws determine a system of screws in space consisting of the "totality of the generators of one system of a skew conicoid which changes in accordance with the variation of a parameter involved in its equation." Professor Hyde assumes a particular arrangement of the determining screws, viz.; they are mutually perpendicular, and have a common point. The envelope of the before-mentioned skew conicoid is found, which will be touched by all the screws of the system. It is a surface of the sixth order whose constants are functions of the differences of the pitches of the determining screws, and is shown to be reducible to a comparatively simple form in which only two of these differences appear. The peculiarities of the surface are discussed, and three diagrams given, illustrating the different forms which it may assume.

In a paper published in the *Mathematische Annalen*, volume 52, pp. 561–581, Professor Dickson showed that the structures of the groups defined by invariants

$$\sum_{i=1}^{m} \lambda_i \xi_i^r \qquad (r > 2)$$

in any Galois field depend upon the structure of the group with the invariant

$$\sum_{i=1}^{m} \xi_2^{p^s+1}$$

in the $GF[p^{2s}]$. The latter group is called the hyperorthogonal group H, since it includes the orthogonal group as a subgroup. It may be represented as a transitive substitution group on N_c symbols

$$\lambda_1\xi_1 + \cdots + \lambda_m\xi_m$$

where $\lambda_1, \cdots, \lambda_m$ are any marks, not all zero, satisfying

$$\lambda_1^{p^s+1} + \cdots + \lambda_m^{p^s+1} = c = \text{constant}.$$

For any $c \neq 0$, we have

$$N_c = p^{s(2m-1)} - (-1)^m p^{s(m-1)};$$

while, for $c = 0$, N_c has the value

$$\omega_m \equiv [p^{ms} - (-1)^m][p^{(m-1)s} - (-1)^{m-1}].$$

The substitutions of H which multiply every index by the same constant form an invariant subgroup of H. The quotient-group $HO(m, p^{2s})$ is a simple group except when $m = 2$, $p^s = 2$; $m = 2$, $p^s = 3$; $m = 3$, $p^s = 2$. Except when $m = 2$, $p^s = 2$, the quotient group may be represented as a transitive substitution group on $N_c \div (p^s + 1)$ letters for $c \neq 0$, as well as upon $\omega_m \div (p^{2s} - 1)$ letters for $c = 0$. The latter number is the smaller except for $p^s = 2$, m even, $m > 2$, when the former number is $\frac{1}{3}(2^{2m-1} - 2^{m-1})$. For $m = 2$, the smaller number is $p^s + 1$, a result in accord with the known isomorphism of $HO(2, p^{2s})$ with the linear fractional group in the $GF[p^s]$.

The paper makes a detailed study of the characteristic equations of hyperorthogonal substitutions. If κ be one root, then is κ^{-p^s} also a root. Inversely, given an equation having this property, we can determine a hyperorthogonal substitution having it as its characteristic equation. Such an equation is never irreducible in the $GF[p^{2s}]$ if $m \neq 3$; it never decompose into linear factors and a single irreducible factor of degree $\neq 3$. If $f_t(\kappa)$ be one irreducible factor, there exists a complementary irreducible factor

$$\kappa^t \{ f_t(\kappa^{-p^{-s}}) \}^{p^s}$$

of equal degree t. Thus, for $m = 4$, if $\kappa^2 - a\kappa + \beta$ be an irreducible factor, then is also

$$\kappa^2 - \kappa a^{p^s} \beta^{-1} + \beta^{-1},$$

where β necessarily belongs to the included $GF[p^s]$. The greater part of the paper is devoted to the reduction of hyperorthogonal substitutions to canonical forms within the group and to their distribution into complete sets of conjugate substitutions. The results are exhaustive for the case $m = 3$. For example, when $p^s = 3$, $m = 3$, the group H is a simple group of order 6048, containing exactly 14 non-conjugate types of substitutions whose periods are 1, 2, 3, 4, 6, 7, 8, 12.

Professor Brown showed that the problem of the motion of a particle in a plane is reducible to the discovery of a certain cylindrical surface whose area between two generating lines is a minimum. A similar result is also shown for the case of a particle whose motion is referred to axes moving in the plane with uniform velocity.

In 1888 (*Proceedings of the Royal Society of Edinburgh*, pp. 99–105) Dr. T. Muir showed that a linear relation exists between certain minors of a centro-symmetric determinant similar to Kronecker's relation between the minors of an axi-symmetric determinant; and in 1900 (same journal, pp. 142–154) he gave two theorems connecting the minors of any determinant, the first of which reduces to Kronecker's relation on imposing the conditions for axi-symmetry, and the second of which reduces to his 1888 relation on imposing the conditions for centro-symmetry. Professor Metzler extends these two theorems given by Dr. Muir so as to include his as special cases and gives a whole series of types of linear relations between the minors of a centro-symmetric determinant, Muir's being one type of the series. The paper also gives the number of relations of each type.

Professor Van Vleck presented a note supplementary to his paper read at the February meeting. The note will be embodied in the paper, which will appear in the *Transactions*.

Professor Osgood's paper, which will appear in the *Transactions*, is in abstract as follows: The integral

$$I = \int_{x_0}^{x_1} F(x, y, y')\,dx$$

or, in the parametric form,

$$I = \int_{t_0}^{t_1} F(x, y, x', y')\,dt$$

is said to be made a minimum by the functions $x = \varphi(t)$, $y = \psi(t)$ corresponding to the curve C, if its value J corresponding to this curve is less than its value I corresponding to any other curve $\overline{C}$ lying in a neighborhood T of C and satisfying the boundary conditions of the problem. Weierstrass has given a sufficient condition for a minimum. The question, however, presents itself: Assuming that Weierstrass's sufficient condition is fulfilled, so that no curve $\overline{C}$, distinct from C, of the class of curves admitted to consideration will give the integral I so small a value as J, may there not still be a set of curves belonging to this class, $\overline{C}_1, \overline{C}_2, \cdots$ which do not cluster about C as their limit, and which have the property that, if $I_1, I_2, \cdots$ denote respectively the values of the integral I formed for these curves,

$$\lim_{n=\infty} I_n = J\,?$$

This question is answered in the negative. It is shown that, if $\mathfrak{T}$ denotes an arbitrarily small neighborhood of C lying wholly within T, the lower limit of I formed for all the curves $\bar{C}$ of the class in question that do not lie wholly within $\mathfrak{T}$ is greater than J by a positive quantity ε, varying for different regions $\mathfrak{T}$, but constant for any given region ; so that

$$I \geqq J + \varepsilon, \qquad \text{when } \bar{C} \text{ does not lie wholly in } \mathfrak{T}.$$

By means of this result it is possible to extend the class of curves $\bar{C}$ with reference to which the integral I possesses the minimum property and thus establish the existence of a more general minimum. Making the generalized definition of the length of a curve his point of departure, Weierstrass inscribed in an arbitrary curve $\bar{C}$ that satisfies the boundary conditions, is continuous, and lies within T, a polygon and formed the sum

$$S = \sum_{i=0}^{n-1} F\left(x_i, y_i, \frac{\Delta x_i}{\Delta t_i}, \frac{\Delta y_i}{\Delta t_i}\right) \Delta t_i = \sum_{i=0}^{n-1} F(x_i, y_i, \Delta x_i, \Delta y_i).$$

If S converges toward one and the same limit, no matter how the vertices of the polygon be chosen, provided merely that all its sides converge toward 0, this limit is taken as the generalized limit $\mathfrak{J}$ and the class of curves admitted to consideration is enlarged so as to include $\bar{C}$. Weierstrass states the theorem that $\mathfrak{J} > J$ if $\bar{C}$ is distinct from C. This theorem is proven in the present paper and it is furthermore shown that

$$\mathfrak{J} \geqq J + \varepsilon \qquad \text{when } \bar{C} \text{ does not lie wholly in } \mathfrak{T}.$$

The theory of the group of euclidean motions yields the notions of ordinary n-dimensional geometry. In his first paper Professor Lovett shows that it can also be made to yield the elements of this geometry. The group is extended relative to the coefficients of the equation of the general quadric. An application of Euler's theorem facilitates the construction of the invariants. In particular it appears that the squares of the semi-axes of the quadric

$$\sum_1^n\sum a_{ij}x_ix_j + 2\sum_{j=1}^n a_{j,\,n+1}x_j + a_{n+1,\,n+1} = 0$$

are given by the equation

$$I_{n+1}^n + \sum_1^{n-1} {}^iI_n^i I_i I_{n+1} x^i + I_n x^n = 0,$$

where

$$I_k = \sum_1^n {}_{i_1 i_2 \cdots i_k} |a_{i_1 i_1}, \ a_{i_2 i_2}, \ \cdots, \ a_{i_k i_k}|.$$

Every space of k dimensions within the n-dimensional space has its own differential geometry. To determine its elements, analogous to those of Gauss for the differential geometry of surfaces in ordinary space, it is only necessary to construct the nth extension of the group of euclidean motions relative to k independent parameters which are assumed to be invariant under the transformations of the group.

Professor Lovett's second paper effects these extensions for the values 1 and $n - 1$ of k, and constructs the corresponding differential invariants. In particular, it appears that the following relations are the generalizations for four dimensional space of the three well-known relations of Gauss, Mainardi, and Codazzi among the Gaussian fundamental quantities of the first and second order :

$$2 \cdot 12_{12} - 11_{22} - 22_{11} = \varphi_1, \quad 2 \cdot 23_{23} - 22_{33} - 33_{22} = \varphi_2,$$

$$2 \cdot 31_{31} - 33_{11} - 11_{33} = \varphi_3,$$

$$12_{33} - 13_{23} = \varphi_4, \quad 13_{22} - 12_{23} = \varphi_5, \quad 13_{12} - 12_{13} = \varphi_6,$$

$$12_2 - 22_1 = \psi_1, \quad 23_3 - 33_2 = \psi_2, \quad 22_3 - 23_2 = \psi_3,$$

$$11_2 - 12_1 = \psi_4, \quad 12_3 - 23_1 = \psi_5, \quad 11_3 - 31_1 = \psi_6,$$

$$12_3 - 31_2 = \psi_7, \quad 33_1 - 31_3 = \psi_8,$$

where

$$\ddot{ij}_{kl} = E_{ij_{u_k u_l}}, \qquad \dot{ij}_k = F_{ij_{u_k}},$$

$$E_{ij} = \sum_1^n {}_k \frac{\partial x_k}{\partial u_i} \frac{\partial x_k}{\partial u_j}, \qquad F_{ij} = \sum_1^n {}_k X_k \frac{\partial^2 x_k}{\partial u_i \partial u_j},$$

$$X_i |E_{11}, \ E_{22}, \ \cdots, \ E_{n-1\, n-1}|^{\frac{1}{2}} = M_i,$$

in which M_i is the determinant formed by suppressing the ith column of the matrix

$$\| x_{1_{u_1}}, \ x_{2_{u_2}}, \ \cdots, \ x_{n_{u_n}} \|,$$

and the φ's and ψ's are functions of the first and second order differential invariants whose forms are easily constructed. The theory of this group gives the whole theory of k-parametric displacements in a space of n-dimensions, as a future note will show.

Dr. Kasner's second paper shows that, in space of n dimensions, the angles of any number of lines less than $n+1$ are all independent; while the $\frac{1}{2}n(n+1)$ angles of $n+1$ lines $L_1, L_2, \cdots, L_{n+1}$ are connected by the single relation

$$\begin{vmatrix} 1 & (1,2) & \ldots\ldots & (1, n+1) \\ (2,1) & 1 & \ldots\ldots & (2, n+1) \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ (n+1, 1) & (n+1, 2) & \ldots\ldots & 1 \end{vmatrix} = 0,$$

where (i, j) denotes the cosine of the angle L_i, L_j. The relations for any number of lines are of the same type. For ordinary space this gives the known relation between the sides and diagonals of a spherical quadrilateral, and the relation between the dihedral angles of a tetrahedron.

Dr. Eisenhart's paper is, in abstract, as follows: An extension of the term isothermal-conjugate system of lines, as given by Bianchi to those systems on a surface of positive curvature for which as parametric curves the second fundamental form reduces to $\lambda(du^2 + dv^2)$, is made to those conjugate systems on surfaces of negative curvature for which this form reduces to $\lambda(du^2 - dv^2)$, where λ is a function of u and v. Results are found for these systems very similar to those given by Bianchi for the former case. A general study is made of these lines on both kinds of surfaces; the determination of these systems on surfaces of positive curvature is effected by methods similar to those used by Bianchi in the study of isothermal orthogonal systems. A geometrical interpretation of isothermal-conjugate systems leads to the theorem that the surfaces obtained by reciprocal radii vectores from surfaces whose lines of curvature are isothermal-conjugate are of the same kind. The greater part of the paper is devoted to a study of surfaces whose lines of curvature are isothermal-conjugate, and it is shown that the general problem of determining such surfaces depends upon the integration of a differential equation of the fourth order, very similar to the equation found by Darboux in his study of isothermic surfaces. It is noted that the surfaces whose lines of curvature are at the same time isothermal and isothermal-conjugate have an isothermal spherical representation, and from this is deduced the theorem that the sphere and minimal surfaces are the only surfaces of constant mean curvature whose lines of curvature are isothermal-conjugate; also that the quadric surfaces have isothermal-conjugate lines of curvature.

It is shown that the plane, sphere, cyclides of Dupin, and surfaces of revolution are the only surfaces which together with their parallels have an isothermal conjugate system of lines of curvature; and that the surfaces of constant total curvature, the surfaces of revolution, and the surfaces whose radii satisfy two definite equations of condition (not given here) are the only Weingarten surfaces possessing the above property. Associated with a surface with isothermal-conjugate systems of lines of curvature is a second surface with isothermal lines of curvature, corresponding to the former by parallelism of tangent planes; it can be found by quadratures. In particular, when the first surface is of constant total curvature, the second is of constant mean curvature and parallel to the former. The paper closes with a discussion of surfaces with plane lines of curvature in both systems which at the same time are isothermal-conjugate. Solutions of this problem are furnished by surfaces of revolution, certain moulded surfaces, cyclides of Dupin, minimal surfaces of Bonnet and Enneper, and other surfaces whose equations are given.

In a paper recently published in the *Transactions*, Dr. Wilczynski has initiated a theory of invariants of a system of linear differential equations. In the present paper, the author considers a remarkable relation between this theory and line geometry. If (y_i) and (z_i) $(i = 1, 2, 3, 4)$ form a fundamental system of simultaneous solutions of the system of differential equations

$$(1)\qquad \begin{aligned} y'' + p_{11}y' + p_{12}z' + q_{11}y + q_{12}z = 0, \\ z'' + p_{21}y' + p_{22}z' + q_{21}y + q_{22}z = 0, \end{aligned}$$

and y_i and z_i be interpreted as the homogeneous coördinates of two points in space, it is seen that (1) defines two curves C_y and C_z as its integral curves. The points (y_i) and (z_i) of these curves, which correspond to the same value of the independent variable x, are joined by a straight line L_{yz}, and these straight lines generate a ruled surface S, called the integrating ruled surface of system (1), which, it is shown, can never be a developable surface.

The most general transformations which convert (1) into another system of the same kind are

$$(2)\qquad x = f(\xi), \quad y = \alpha\eta + \beta\zeta, \quad z = \gamma\eta + \delta\zeta,$$

where f, α, β, γ, δ are arbitrary functions of ξ. It is noted that these transformations leave invariant the ruled surface