i. e., the case in which the function φ vanishes at one or both ends of the interval need not be excluded. The interval on the t-axis would, however, then extend to infinity in one or both directions, and the fundamental theorem concerning equation (2) from which we started would on longer be sufficient, but would have to be replaced by a theorem which states that, if $q \leqq 0$, no solution of (2) which vanishes at a finite point can approach a finite limit as x becomes either positively or negatively infinite, and that no solution of (2) can approach finite limits both when $x = + \infty$ and when $x = - \infty$.

The extension which our other theorems gain by the use of (7′) in place of (7) is easily seen. In using functions φ which vanish at one of the ends of the interval it is useful to know that if φ' also vanishes then φ cannot possibly satisfy (8),—a fact whose proof we also omit.

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CONCERNING REAL AND COMPLEX CONTINUOUS GROUPS.

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1. THIS paper aims to illustrate certain differences and certain analogies between related real and complex continuous groups. Lie's theory has been developed chiefly for the latter groups, the modifications necessary for real groups being treated quite briefly.

In §§ 2–4 are exhibited a real group in m variables and a real group in $2m$ variables, each of m^2 parameters, such that the corresponding complex groups are of like structure. In §§ 5–8, it is shown for $m = 2$ that the two real groups have different structures. Of the three proofs given, the first two are analytic and involve little technical knowledge of group theory, while the third group is geometric and gives a better insight into the nature of the question.

In § 10, it is illustrated for the case $m = 2$ how the general m-ary linear homogeneous complex continuous group gives rise to an isomorphic $2m$-ary linear homogeneous real continuous group. Similarly, the complex projective groups lead to groups of birational quadratic transformations.

The investigation has direct contact with the author's determination * of the structure of the largest group in the $GF[p^{2n}]$ leaving invariant $\xi_1\bar{\xi}_1 + \xi_2\bar{\xi}_2 + \cdots + \xi_m\bar{\xi}_m$, where $\bar{\xi}_i$ is conjugate to ξ_i with respect to the $GF[p^n]$; also with the paper by Moore † on the universal invariant of finite groups of linear substitutions.

2. Consider the group G_m of all substitutions

$$S: \qquad \xi_i' = \sum_{j=1}^{m} a_{ij}\xi_j \qquad (i = 1, \cdots, m),$$

the coefficients and variables being complex numbers, such that S leaves formally invariant the Hermitian form

$$\Phi \equiv \xi_1\bar{\xi}_1 + \xi_2\bar{\xi}_2 + \cdots + \xi_m\bar{\xi}_m.$$

The conditions upon the coefficients are seen to be

$$(1) \qquad \sum_{i=1}^{m} a_{ij}\bar{a}_{ij} = 1, \quad \sum_{i=1}^{m} a_{ij}\bar{a}_{ik} = 0 \qquad (j, k = 1, \cdots, m; j \neq k).$$

It follows that the inverse of S has the form

$$S^{-1}: \qquad \xi_i' = \sum_{j=1}^{m} \bar{a}_{ji}\xi_j \qquad (i = 1, \cdots, m).$$

The group G_m is evidently continuous. To obtain the general infinitesimal transformation, set $I^2 = -1$ and

$$a_{ii} = 1 + (\alpha_{ii} + I b_{ii})\,\delta t, \quad a_{ij} = (\alpha_{ij} + I b_{ij})\,\delta t$$
$$(i, j = 1, \cdots, m; j \neq i).$$

Substituting these values in the relations (1) and retaining only the first power of δt, we find that

$$1 + 2\alpha_{jj}\delta t = 1, \quad (\alpha_{jk} + \alpha_{kj}) + I(b_{kj} - b_{jk}) = 0$$
$$(j, k = 1, \cdots, m; j \neq k).$$

The conditions upon the general infinitesimal transformation

$$(2) \qquad \delta\xi_i \equiv \xi_i' - \xi_i = \sum_{j=1}^{m} (\alpha_{ij} + I b_{ij})\,\delta t \cdot \xi_j$$

are therefore the following

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* *Math. Annalen*, vol. 52, pp. 561–581.

† *Math. Annalen*, vol. 50, pp. 213–219.

$$(3) \qquad a_{jk} = -a_{kj}, \quad b_{jk} = b_{kj} \qquad (j, k = 1, \cdots, m).$$

The general infinitesimal transformation of G_m is therefore a linear combination with *real* constant coefficients of

$$(4) \qquad \begin{gathered} B_{ii}' \equiv I\xi_i \frac{\partial f}{\partial \xi_i}, \\ B_{ij}' \equiv I\xi_j \frac{\partial f}{\partial \xi_i} + I\xi_i \frac{\partial f}{\partial \xi_j}, \quad A_{ij}' \equiv \xi_j \frac{\partial f}{\partial \xi_i} - \xi_i \frac{\partial f}{\partial \xi_j}. \end{gathered}$$

Here B_{ij}' was obtained from (3) by setting $b_{ij} = b_{ji} = 1$ and the remaining constants all zero; A_{ij}' by setting $a_{ij} = -a_{ji} = 1$ and the remaining coefficients all zero.

The number of linearly independent transformations (4) is evidently m^2. If *complex* multipliers were allowed, we could derive from (4) the m^2 transformations

$$\xi_j \frac{\partial f}{\partial \xi_i} \qquad (i, j = 1, \cdots, m),$$

and therefore the general transformation of the m-ary linear homogeneous continuous group.

3. We obtain a continuous group R_{2m} on $2m$ real variables with real coefficients by replacing ξ_i by $X_i + IY_i$ for $i = 1, \cdots, m$ and separating reals and pure imaginaries. Relation (2) gives

$$X_i' + IY_i' - X_i - IY_i = \sum_{i=1}^{m} \{ (a_{ij}X_j - b_{ij}Y_j) + I(a_{ij}Y_j + b_{ij}X_j) \} \delta t.$$

Hence the general infinitesimal transformation of R_{2m} is

$$(5) \qquad \delta X_i = \sum_{j=1}^{m} (a_{ij}X_j - b_{ij}Y_j)\delta t, \quad \delta Y_i = \sum_{j=1}^{m} (b_{ij}X_j + a_{ij}Y_j) \qquad (i = 1, \cdots, m).$$

Denote by B_{ij} the transformation obtained by setting $b_{ij} = b_{ji} = 1$ and the remaining coefficients equal to zero; by A_{ij} that obtained by setting $a_{ij} = -a_{ji} = 1$ and the remaining coefficients equal to zero. Employing the usual abbreviations $p_i \equiv \frac{\partial f}{\partial X_i}$, $q_i \equiv \frac{\partial f}{\partial Y_i}$, we have

$$B_{ii} \equiv X_i q_i - Y_i p_i, \quad B_{ij} \equiv X_j q_i - Y_j p_i + X_i q_j - Y_i p_j,$$
$$A_{ij} \equiv Y_j q_i + X_j p_i - Y_i q_j - X_i p_j.$$

Since $B_{ij} \equiv B_{ji}$, $A_{ij} \equiv -A_{ji}$, there are exactly m^2 independent transformations, from which the general infinitesimal transformation of R_{2m} may be derived as a linear expression with real coefficients. In view of the identity

$$\Phi \equiv \sum_{i=1}^{m} \xi_i \bar{\xi}_i = \sum_{i=1}^{m} (X_i^2 + Y_i^2)$$

the group R_{2m} is an orthogonal group. As a check it may be verified that the transformations B_{ij}, A_{ij} leave Φ absolutely invariant.

4. The following commutator (Klammerausdruck) relations are readily formed :

$$(6) \quad \begin{array}{l} (B_{ii}B_{jj})=0, \ (B_{ii}B_{ij})=A_{ij}, \ (B_{ii}A_{ij})=-B_{ij}, \ (B_{ij}B_{ik})=A_{jk}, \\ (A_{ij}B_{ik})=B_{kj}, \quad (A_{ij}A_{ik})=A_{jk}, \quad (B_{ij}A_{ij})=2B_{ii}-2B_{jj}, \end{array}$$

for $i, j, k = 1, \cdots, m$, with i, j, k distinct. If both subscripts of one symbol be different from the subscripts of the other, their commutator is zero.

It is readily verified that the transformations B_{ij}', A_{ij}' of §2 satisfy the commutator relations (6). This property would be expected to follow from the connection between G_m and R_{2m}. We may conclude that the continuous group with complex coefficients which is generated by the transformations B_{ij}, A_{ij} is isomorphic with the continuous complex group generated by B_{ij}', A_{ij}'.

Denote by B_{ij}'' the symbol obtained upon dropping the factor I from the symbol B_{ij}'. In the domain of real numbers, the transformations B_{ij}'', A_{ij}' generate the continuous group G_m' of all real linear homogeneous transformations in m variables. The symbols B_{ij}'', A_{ij}' do not satisfy the commutator relations (6). It is shown in §§ 5–8 that there does not exist in the real group G_2' $(m = 2)$ a set of four independent infinitesimal transformations which satisfy the commutator relations (6), so that G_2' and R_4 are nonisomorphic real continuous groups of four parameters each.

5. For $m = 2$, the relations (6) are the following :

$$(B_{11}B_{22}) = 0, \quad (B_{11}B_{12}) = A_{12}, \quad (B_{11}A_{12}) = -B_{12},$$

$$(B_{22}B_{12}) = -A_{12}, \quad (B_{22}A_{12}) = B_{12}, \quad (B_{12}A_{12}) = 2B_{11} - 2B_{22}.$$

The first derived group is therefore the three-parameter group generated by A_{12}, B_{12}, $B_{11} - B_{22}$. The only (ausgezeichnete) transformation whose commutator with B_{11}, B_{22}, B_{12}, A_{12} is zero is seen to be $B_{11} + B_{22}$, aside from a constant factor. Hence, if R_4 be isomorphic with G_2', $B_{11}+B_{22}$ must correspond with $\xi_1\frac{\partial f}{\partial\xi_1}+\xi_2\frac{\partial f}{\partial\xi_2}$ and the above three-parameter group with the first derived group of G_2'. To normalize R_4, set

$$Z_1\equiv\tfrac{1}{2}(B_{11}-B_{22}),\quad Z_2\equiv\tfrac{1}{2}A_{12},\quad Z_3\equiv-\tfrac{1}{2}B_{12},\quad Z_4\equiv B_{11}+B_{22}.$$

The above commutator relations then give

$$(7)\qquad (Z_1Z_2)=Z_3,\quad (Z_2Z_3)=Z_1,\quad (Z_3Z_1)=Z_2,$$

$$(8)\qquad (Z_4Z_1)=0,\quad (Z_4Z_2)=0,\quad (Z_4Z_3)=0.$$

The first derived group of G_2' is generated by

$$V_1\equiv\xi_1\frac{\partial f}{\partial\xi_2},\quad V_2=\xi_1\frac{\partial f}{\partial\xi_1}-\xi_2\frac{\partial f}{\partial\xi_2},\quad V_3\equiv\xi_2\frac{\partial f}{\partial\xi_1},$$

subject to the commutator relations

$$(9)\ (V_1V_2)=-2V_1,\quad (V_2V_3)=-2V_3,\quad (V_3V_1)=-V_2.$$

To establish the non-isomorphism of R_4 and G_2', it suffices to prove that their first derived groups are non-isomorphic when considered as *real* continuous groups.

6. The most natural method of proof consists in showing that it is impossible to determine linear combinations of V_1, V_2, V_3 with real constant coefficients

$$Z_i'\equiv a_iV_1+b_iV_2+c_iV_3\qquad (i=1,2,3),$$

of determinant $\Delta\equiv\sum a_1b_2c_3\neq 0$, which satisfy relations (7). We observe that

$$(Z_1'Z_2')\equiv-2V_1(a_1b_2-b_1a_2)+V_2(a_1c_2-c_1a_2)-2V_3(b_1c_2-c_1b_2).$$

The conditions that the right member shall equal Z_3' are

$$(10)\ a_1b_2-b_1a_2=-\tfrac{1}{2}a_3,\quad a_1c_2-c_1a_2=b_3,\quad b_1c_2-c_1b_2=-\tfrac{1}{2}c_3.$$

By advancing the subscripts of a_i, b_i, c_i, we obtain the conditions for the identities $(Z_2'Z_3')=Z_1'$, $(Z_3'Z_1')=Z_2'$

$$(11)\ a_2b_3-b_2a_3=-\tfrac{1}{2}a_1,\ a_2c_3-c_2a_3=b_1,\ b_2c_3-c_2b_3=-\tfrac{1}{2}c_1,$$

$$(12)\ a_3b_1-b_3a_1=-\tfrac{1}{2}a_2,\ a_3c_1-c_3a_1=b_2,\ b_3c_1-c_3b_1=-\tfrac{1}{2}c_2.$$

In view of the relations (11) and (12),

$$-\tfrac{1}{2}\begin{vmatrix}a_1 & b_1\\ a_2 & b_2\end{vmatrix}=a_3\Delta.$$

Applying the first relation (10), $\tfrac{1}{4}a_3=a_3\Delta$. In a similar manner, or by advancing the subscripts (which does not alter Δ), we find

$$\tfrac{1}{4}a_1 = a_1\Delta, \quad \tfrac{1}{4}a_2 = a_2\Delta.$$

Since a_1, a_2, a_3 are not all zero, it follows that $\Delta = \frac{1}{4}$.

Multiplying equations (12) by c_2, $-b_2$, a_2 respectively and adding the resulting equations, we find

$$(13) \qquad \Delta = -b_2^2 - a_2c_2.$$

Employing the multipliers c_1, $-b_1$, a_1, we find

$$(14) \qquad 0 = -b_1b_2 - \tfrac{1}{2}a_2c_1 - \tfrac{1}{2}a_1c_2.$$

In a similar manner, or by advancing the subscripts,

$$(15) \qquad \Delta = -b_1^2 - a_1c_1.$$

By (14) and the second equation of set (10),

$$b_3^2 = (a_1c_2 - c_1a_2)^2 = 4b_1^2b_2^2 - 4a_1c_1a_2c_2.$$

Eliminating a_1c_1 and a_2c_2 by (15) and (13), and setting $\Delta = \frac{1}{4}$,

$$b_1^2 + b_2^2 + b_3^2 = -\tfrac{1}{4}.$$

But this equation is impossible for real values of the b_i.

7. A second proof is derived from the following investigation which gives certain interesting properties of the group Γ generated by Z_1, Z_2, Z_3, subject to the relations (7). Set

$$\begin{aligned} Z_1 &= \alpha_1 U_1 + \alpha_2 U_2 + \alpha_3 U_3, \\ Z_2 &= \beta_1 U_1 + \beta_2 U_2 + \beta_3 U_3, \\ Z_3 &= \gamma_1 U_1 + \gamma_2 U_2 + \gamma_3 U_3, \end{aligned} \qquad \Delta \equiv \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix} \neq 0.$$

We obtain by solution the most general set of independent infinitesimal transformations U_1, U_2, U_3, of the group Γ. We seek the commutator relations of the U_i. Denote by α_i' the first minor (without prefixed sign) of α_i in Δ, β_i' the first minor of β_i, γ_i' the first minor of γ_i. Form (Z_2Z_3) and equate the result to Z_1; expand similarly $(Z_3Z_1) = Z_2$, $(Z_1Z_2) = Z_3$. The results are

$$\begin{aligned} Z_1 &= \alpha_1'(U_2U_3) - \alpha_2'(U_3U_1) + \alpha_3'(U_1U_2), \\ Z_2 &= -\beta_1'(U_2U_3) + \beta_2'(U_3U_1) - \beta_3'(U_1U_2), \\ Z_3 &= \gamma_1'(U_2U_3) - \gamma_2'(U_3U_1) + \gamma_3'(U_1U_2). \end{aligned}$$

The determinant of the coefficients equals Δ^2, being equal to the determinant of the first minors of Δ. Moreover,

$$\begin{vmatrix} \beta_2' & \beta_3' \\ \gamma_2' & \gamma_3' \end{vmatrix} = \alpha_1\Delta.$$

The solution of the above relations therefore gives

$$\Delta(U_2U_3) = \alpha_1Z_1 + \beta_1Z_2 + \gamma_1Z_3,$$
$$\Delta(U_3U_1) = \alpha_2Z_1 + \beta_2Z_2 + \gamma_2Z_3,$$
$$\Delta(U_1U_2) = \alpha_3Z_1 + \beta_3Z_2 + \gamma_3Z_3.$$

The matrix of the coefficients on the right is the transposed of the matrix of the coefficients of the U_i in the expressions for the Z_i. Eliminating the Z_i, we have

$$\begin{array}{r|lll} & U_1 & U_2 & U_3 \\ \hline \Delta(U_2U_3)= & \alpha_1^2+\beta_1^2+\gamma_1^2 & \alpha_1\alpha_2+\beta_1\beta_2+\gamma_1\gamma_2 & \alpha_1\alpha_3+\beta_1\beta_3+\gamma_1\gamma_3 \\ \Delta(U_3U_1)= & \alpha_1\alpha_2+\beta_1\beta_2+\gamma_1\gamma_2 & \alpha_2^2+\beta_2^2+\gamma_2^2 & \alpha_2\alpha_3+\beta_2\beta_3+\gamma_2\gamma_3 \\ \Delta(U_1U_2)= & \alpha_1\alpha_3+\beta_1\beta+\gamma_1\gamma_3 & \alpha_2\alpha_3+\beta_2\beta_3+\gamma_2\gamma_3 & \alpha_3^2+\beta_3^2+\gamma_3^2 \end{array}$$

The symmetry of the matrix of coefficients is in accord with a known property of the group.*

In order that the transformations U_i should satisfy the same commutator relations (9) as the transformations V_i it is necessary that $(U_2U_3) = -2U_3$, so that $\alpha_1^2 + \beta_1^2 + \gamma_1^2 = 0$. For real values of $\alpha_1, \beta_1, \gamma_1$, this requires $\alpha_1 = \beta_1 = \gamma_1 = 0$, contrary to hypothesis. Hence the real group Γ of the Z_i is not isomorphic with the real group of the V_i.

To obtain the most general set of three infinitesimal transformations U_i of Γ which satisfy the same commutator relations (7) as the transformations Z_i themselves,

$$(U_2U_3) = U_1, \quad (U_3U_1) = U_2, \quad (U_1U_2) = U_3,$$

it is necessary and sufficient to take solutions $\alpha_i, \beta_i, \gamma_i$ of

$$\Delta = \alpha_1^2 + \beta_1^2 + \gamma_1^2, \quad \Delta = \alpha_2^2 + \beta_2^2 + \gamma_2^2, \quad \Delta = \alpha_3^2 + \beta_3^2 + \gamma_3^2.$$
$$0 = \alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2, \quad 0 = \alpha_1\alpha_3 + \beta_1\beta_3 + \gamma_1\gamma_3,$$
$$0 = \alpha_2\alpha_3 + \beta_2\beta_3 + \gamma_2\gamma_3.$$

These are the conditions for an orthogonal substitution, the invariant relation being

$$Z_1^2 + Z_2^2 + Z_3^2 \equiv \Delta\,(U_1^2 + U_2^2 + U_3^2).$$

8. To give a third proof, based upon geometric considerations, it suffices to consider the adjoint groups of the three-parameter groups in question. The adjoint of the group of the V_i is

$$2e_2\frac{\partial f}{\partial e_1} - e_3\frac{\partial f}{\partial e_2}, \quad -2e_1\frac{\partial f}{\partial e_1} + 2e_3\frac{\partial f}{\partial e_3}, \quad e_1\frac{\partial f}{\partial e_2} - 2e_2\frac{\partial f}{\partial e_3},$$

having as its only invariant curve the real conic

*Lie-Scheffers, Vorlesungen über continuierliche Gruppen, p. 567.

(16) $$e_1e_3 + e_2^2 = 0.$$

The adjoint group of the group of the Z_i is

$$e_3\frac{\partial f}{\partial e_2} - e_2\frac{\partial f}{\partial e_3}, \quad -e_3\frac{\partial f}{\partial e_1} + e_1\frac{\partial f}{\partial e_3}, \quad e_2\frac{\partial f}{\partial e_1} - e_1\frac{\partial f}{\partial e_2},$$

having as its only invariant the imaginary conic

(17) $$e_1^2 + e_2^2 + e_3^2 = 0.$$

Now the replacement of one complete set of independent infinitesimal transformationsof a group by a second complete set merely gives rise to a linear homogeneous transformation upon the variables e_i of the adjoint group. The latter will be a real transformation if the second set is expressed in terms of the first by real coefficients. Since the equations (16) and (17) can not be transformed into each other by a real ternary substitution, it follows that the transformations V_i are not expressible as real linear functions of the Z_i.

The method of reduction of three-parameter non-integrable groups to a normal type given in Lie-Scheffers, Vorlesungen, pp. 566–568 is immediately applicable only to complex groups. For real groups there are two (and, indeed, only two*) distinct cases, according as the invariant conic (necessarily non-degenerate) is real or imaginary. The two methods there given as optional for complex groups are to be differentiated for real groups to correspond to the cases of real and imaginary conics, yielding respectively the normal types (I) and (I′) of p. 568, or types (9) and (7) respectively of this paper.

9. The real four-parameter groups G_2' and R_4 have been proved to have different structures. Applying the imaginary transformation of variables

$$X_1 = x_1, \quad Y_1 = y_1, \quad X_2 = Ix_2, \quad Y_2 = Iy_2 \qquad (I^2 = -1),$$

the infinitesimal transformations B_{11}, B_{22}, B_{12}, A_{12} of R_4 become

$$b_{11} \equiv x_1\frac{\partial f}{\partial y_1} - y_1\frac{\partial f}{\partial x_1}, \quad b_{22} \equiv x_2\frac{\partial f}{\partial y_2} - y_2\frac{\partial f}{\partial x_2},$$

$$b_{12} \equiv x_2\frac{\partial f}{\partial y_1} - y_2\frac{\partial f}{\partial x_1} - x_1\frac{\partial f}{\partial y_2} + y_1\frac{\partial f}{\partial x_2},$$

$$a_{12} \equiv y_2\frac{\partial f}{\partial y_1} + x_2\frac{\partial f}{\partial x_1} + y_1\frac{\partial f}{\partial y_2} + x_1\frac{\partial f}{\partial x_2}.$$

* An irreducible ternary quadratic form with real coefficients is reducible either to $b(e_1^2 + e_2^2 + e_3^2)$ or to $b(e_1^2 + e_2^2 - e_3^2)$.

They satisfy the commutator relations

$$(b_{11}b_{22}) = 0, \quad (b_{11}b_{12}) = a_{12}, \ (b_{11}a_{12}) = -b_{12},$$
$$(b_{22}b_{12}) = -a_{12}, \ (b_{22}a_{12}) = b_{12}, \ (b_{12}a_{12}) = -2b_{11} + 2b_{22}.$$

Except for the last relation, these are identical with the commutator relations of B_{11}, B_{12}, B_{22}, A_{12} (§5). Setting

$$W_1 \equiv a_{12}, \ W_2 \equiv -b_{12}, \ W_3 \equiv b_{11} - b_{22}, \ W_4 \equiv b_{11} + b_{22},$$

we have the commutator relations

$$(W_1W_2) = -2W_3, \ (W_2W_3) = 2W_1, \ (W_3W_1) = 2W_2,$$
$$(W_4W_1) = 0, \quad (W_4W_2) = 0, \quad (W_4W_3) = 0.$$

These relations are also satisfied by the transformations

$$w_1 \equiv xq + yp, \ w_2 \equiv xp - yq, \ w_3 \equiv xq - yp, \ w_4 \equiv xp + yq,$$

which generate the general binary group G_2'. *By an imaginary transformation of variables, R_4 may be given a real form having the same structure as G_2'.*

10. Consider the group G of all binary transformations

$$S: \quad \begin{cases} X' = \alpha X + \gamma Y \\ Y' = \beta X + \delta Y \end{cases} \qquad (\alpha\delta - \beta\gamma = 1)$$

upon complex variables X, Y with complex coefficients of determinant unity. Let $I^2 = -1$ and set

$$X = x + Ix_1, \ Y = y + Iy_1, \ \alpha = a + Ia_1, \ \beta = b + Ib_1, \text{ etc.}$$

Then S corresponds to the quaternary transformation

Σ:

	x	x_1	y	y_1
$x' =$	a	$-a_1$	c	$-c_1$
$x_1' =$	a_1	a	c_1	c
$y' =$	b	$-b_1$	d	$-d_1$
$y_1' =$	b_1	b	d_1	d

The relation $\alpha\delta - \beta\gamma = 1$ gives

$$ad - bc - a_1d_1 + b_1c_1 = 1, \quad ad_1 + a_1d - bc_1 - b_1c = 0.$$

The determinant of Σ is seen to equal

$$(ad - bc - a_1d_1 + b_1c_1)^2 + (ad_1 + a_1d - bc_1 - b_1c)^2 = 1.$$

To the product S_1S_2 of two substitutions of the form S, corresponds the product $\Sigma_1\Sigma_2$ of the corresponding substitutions Σ. Hence the group G is isomorphic with the group Γ of the substitution $\sum$. But $\sum$ reduces to the identity only when S is the identity. Hence the isomorphism is holoedric.

By the usual method, the general infinitesimal transformation of Γ is found to be a linear combination of the following linearly independent transformations :

	$\frac{\partial f}{\partial x}$	$\frac{\partial f}{\partial x_1}$	$\frac{\partial f}{\partial y}$	$\frac{\partial f}{\partial y_1}$
A	x	x_1	$-y$	$-y_1$
A_1	$-x_1$	x	y_1	$-y$
B	0	0	x	x_1
B_1	0	0	$-x_1$	x
C	y	y_1	0	0
C_1	$-y_1$	y	0	0

The real group Γ therefore possesses no invariant. The non-vanishing second minors of the matrix of coefficients are

$$P \equiv x^2 + x_1^2, \quad Q \equiv y^2 + y_1^2, \quad R = xy + x_1y_1, \quad S = x_1y - xy_1.$$

Upon them the group Γ gives rise to the following transformations :

	$\frac{\partial f}{\partial P}$	$\frac{\partial f}{\partial Q}$	$\frac{\partial f}{\partial R}$	$\frac{\partial f}{\partial S}$
A	$2P$	$-2Q$	0	0
A_1	0	0	$-2S$	$2R$
B	0	$2R$	P	0
B_1	0	$-2S$	0	$-P$
C	$2R$	0	Q	0
C_1	$2S$	0	0	Q

The determinants of the fourth order of this matrix are all identically zero. To obtain the homogeneous invariants, we annex Euler's homogeneous operator

$$H \equiv P\frac{\partial f}{\partial P} + Q\frac{\partial f}{\partial Q} + R\frac{\partial f}{\partial R} + S\frac{\partial f}{\partial S}.$$

The determinant of the coefficients of A, A_1, B, H equals $8PR(S^2+R^2-PQ)$. The determinant of A, A_1, B_1, H equals $-8PS(S^2+R^2-PQ)$. In this way the only homogeneous invariant is seen to be

$$\psi \equiv S^2 + R^2 - PQ.$$

In terms of the initial variables x, x_1, y, y_1, we see that ψ vanishes identically. Also

$$P \equiv |x + Ix_1| \equiv |X|, \quad Q \equiv |Y|,$$

$$\frac{Y}{X} \equiv \frac{R - IS}{P}, \quad \frac{X}{Y} \equiv \frac{R + IS}{Q}.$$

The group G is hemiedrically isomorphic with the group of linear fractional substitutions

$$(18) \qquad Z' = \frac{\alpha + \gamma Z}{\beta + \delta Z'}, \quad Z \equiv \frac{Y}{X}.$$

The quaternary group on P, Q, R, S is isomorphic with a ternary fractional group on Q/P, R/P, S/P. But

$$\frac{Q}{P} \equiv \left(\frac{R}{P}\right)^2 + \left(\frac{S}{P}\right)^2.$$

Eliminating Q/P, we obtain a group of birational quadratic transformations in the plane. It may evidently be obtained more directly from the transformations (18).

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ON HOLOMORPHISMS AND PRIMITIVE ROOTS.

BY DR. G. A. MILLER.

(Read before the American Mathematical Society, February 23, 1901.)

IN an earlier note* it was observed that every holomorphism of an abelian group with itself can be obtained by establishing an isomorphism between the abelian group and one of its subgroups (which may sometimes be the entire group) and associating the product of corresponding operators with the original operator of the group. The present note is devoted to some additional developments along this line and especially to some elementary results in the theory of numbers which may be derived by this method.

Let s_1 represent an operator of order p^m (p being any prime number) and let P, the group generated by s_1, be

* BULLETIN, Vol. 6 (1900), p. 337.