# NON-OSCILLATORY LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER.

BY PROFESSOR MAXIME BÔCHER.

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WE shall be concerned with the differential equation

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0, \tag{1}$$

and for the sake of simplicity we will assume that the coefficients $p$ and $q$ are, throughout the finite interval $a \leqq x \leqq b$, continuous real functions of the real variable $x$. We shall find it convenient to lay down the following definition :

*The equation (1) is said to be oscillatory or non-oscillatory in the interval $a \leqq x \leqq b$ according as it does or does not have at least one solution (not identically zero) which vanishes more than once in this interval.*

It is my object in the present paper to deduce certain conditions (chiefly sufficient conditions) that the equation (1) should be non-oscillatory. Such conditions have been obtained by Picard (Traité d'analyse, volume III, pp. 101–104); but the method which I use is not only entirely different and, as it seems to me, less artificial than that of Picard, but yields, besides all of Picard's results, others which Picard's method does not give.

My starting point is the special case $p = 0$ :

$$\frac{d^2y}{dx^2} + qy = 0. \tag{2}$$

*Equation (2) is non-oscillatory in the interval $a \leqq x \leqq b$ if throughout this interval $q \leqq 0$.*

For if (2) has a solution $y$ which vanishes more than once in the interval in question, let $x_1$ and $x_2$ be two successive roots of $y$. We may, without loss of generality, assume $y > 0$ when $x_1 < x < x_2$, as, if this were not the case, we could replace $y$ by $y_1 = -y$. We have then $y'(x_1) > 0$, $y'(x_2) < 0$; but by the law of the mean

$$y'(x_2) - y'(x_1) = (x_2 - x_1)\, y''(\xi) \qquad (x_1 < \xi < x_2).$$

Accordingly $y''(\xi) < 0$. This, however, is impossible, since by equation (2) $y''(\xi) = -q(\xi).y(\xi)$.

By reducing (1) to the binomial form (2), we shall get a theorem concerning (1) similar to the one just proved for (2). This reduction is most commonly performed by means of a change of dependent variable $y$.* It can, however, equally well be performed by changing the independent variable $x$, or, more generally, by changing both independent and dependent variable. We will consider at once this general transformation

$$(3) \qquad t = f(x), \quad y = \varphi(x)\,\bar{y}.$$

We assume here that $f$ and $\varphi$ have continuous first and second derivatives throughout the interval $a \leqq x \leqq b$. Furthermore, since we do not wish the solutions of the transformed equation to become infinite, we assume that $\varphi$ does not vanish, say for distinctness

$$\varphi(x) > 0 \qquad (a \leqq x \leqq b).$$

Finally, since we wish the interval $a \leqq x \leqq b$ to correspond in a one to one manner to an interval on the $t$-axis, we assume that $f'$ does not vanish in the interval $a \leqq x \leqq b$.

A peculiarity of this transformation which makes it available for our purposes may be stated as follows:

*The oscillatory or non-oscillatory character of equation* (1) *is invariant with regard to transformations* (3).

The transformation (3) carries (1) over into

$$(4) \quad \frac{d^2\bar{y}}{dt^2} + \frac{1}{f'}\left[\frac{f''}{f'} + 2\frac{\varphi'}{\varphi} + p\right]\frac{d\bar{y}}{dt} + \frac{\varphi'' + p\varphi' + q\varphi}{\varphi \cdot (f')^2}\,\bar{y} = 0,$$

accents denoting differentiation with regard to $x$.

Choosing $\varphi$ at pleasure, subject to the restrictions above mentioned, let us determine $f$ so that the second term of (4) drops out,

$$(5) \qquad f = k\int_c^x \varphi^{-2} e^{-\int_c^x p\,dx}\,dx + k' \qquad (k \neq 0),$$

where $c$ is any point of the interval $a \leqq x \leqq b$.

It is important to notice that all the conditions which $f$ was to fulfill are satisfied by this function.

Equation (4) now reduces to

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*See, *e. g.*, Forsyth's Treatise on differential equations, p. 88. I note in passing that this reduction is possible only if the coefficient $p$ has a continuous first derivative, a restriction which need not be imposed if we reduce to the binomial form by a change of independent variable.

$$(6)\qquad \frac{d^2\bar{y}}{dt^2}+\frac{\varphi^3}{k^2}e^{2\int_c^x p\,dx}(\varphi''+p\varphi'+q\varphi)\bar{y}=0.$$

When we apply to this equation the theorem proved concerning equation (2), we see that equation (6), and accordingly also equation (1), will be non-oscillatory provided $\varphi''+p\varphi'+q\varphi\leqq 0$. That is

*If a function $\varphi$ exists which, together with its first and second derivatives, is continuous throughout the interval $a\leqq x\leqq b$, and which satisfies the two conditions*

$$(7)\qquad \varphi>0 \qquad (a\leqq x\leqq b),$$

$$(8)\qquad \varphi''+p\varphi'+q\varphi\leqq 0 \qquad (a\leqq x\leqq b),$$

*then* (1) *is non-oscillatory in this interval.**

By assuming for $\varphi$ special functions we can obtain useful and easily applied criteria for proving that special equations of the form (1) are non-oscillatory. We add a few such criteria, noting to the left the function $\varphi$ used, and to the right the special conditions, if any, which must be satisfied if the formula is to be applied. In these formulæ $m$ and $a$ denote constants to which we may assign any real values we please.

(*a*) $\varphi=1,\qquad q\leqq 0.$

(*b*) $\varphi=e^{mx},\qquad q\leqq -mp-m^2.$

(*c*) $\varphi=x^{-m},\qquad q\leqq \dfrac{mxp-m(m+1)}{x^2}\qquad (0<a).$

(*d*) $\varphi=m-x,\qquad q\leqq \dfrac{p}{m-x}\qquad (b<m).$

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* By letting $\phi=e^{\int_c^x \lambda dx}$ we obtain the following theorem :

*If a function $\lambda$ exists which, together with its first derivative, is continuous throughout the interval $a\leqq x\leqq b$, and which satisfies the condition*

$$(8')\qquad q\leqq \lambda'-\lambda^2+\lambda p \qquad (a\leqq x\leqq b),$$

*then* (1) *is non-oscillating in this interval.*

Conversely, the theorem of the text follows from this one ; so that the two theorems are precisely equivalent to each other.

Picard (l. c., p. 102) deduces by another method a theorem identical with the one just stated, except that the inequality (8′) is replaced by

$$q\leqq \lambda'-(\lambda-p/2)^2.$$

This inequality is always satisfied when (8′) is satisfied, but the converse is not true. It is only when $p=0$ that Picard's result is as general as ours.

$(e) \quad \varphi = m^2 - x^2, \qquad q \leqq \frac{2(1+xp)}{m^2 - x^2} (-|m| < a < b < |m|).$

$(f) \quad \varphi = \sin m(x - \alpha), \ q \leqq m^2 - mp \operatorname{ctn} m(x - \alpha)$

$$\left(\alpha < a < b < \alpha + \frac{\pi}{m}\right) \quad (m > 0).$$

$(g)^* \quad \varphi = e^{-m\int p dx} \qquad q \leqq m(1-m)p^2 + mp'$ (provided $p$ has a continuous derivative).

Formulæ of this sort might of course be multiplied indefinitely. To show how they are to be applied let us consider the simplest case of Bessel's equation

$$(9) \qquad y'' + \frac{1}{x} y' + y = 0.$$

Since $p$ is here discontinuous at the point $x = 0$, we can consider only intervals which do not include, or even reach up to this point. Since the equation is unchanged by replacing $x$ by $-x$, it will be sufficient to consider intervals in which $x$ is positive.

Formulæ $(a)$, $(c)$, $(g)$ yield us no information whatever with regard to this equation. Formula $(b)$ is most serviceable here if we let $m = -1$. It then shows us that (9) is non-oscillatory in the interval $a \leqq x \leqq \frac{1}{2}$, where $a$ is any small positive quantity. If we let $m = 2$ in formulæ $(d)$ and $(e)$, they each show us that (9) is non-oscillatory in the interval $a \leqq x \leqq b$, when $0 < a < b < 2$. This is the best result these two formulæ can be made to yield if we wish to consider an interval starting from a point arbitrarily near the point $x = 0$. Of the seven formulæ written above, $(f)$ gives the best result when applied to intervals of the sort just described, since when we let $m = \frac{1}{2}\sqrt{2}$, $\alpha = -\frac{\pi}{2m}$, it shows us that (9) is non-oscillatory in the interval $a \leqq x \leqq b$ when

$$0 < a < b < \frac{\pi\sqrt{2}}{2} = 2.22.$$

This result, as we shall see in a moment, is nearly as good as any method could give us.

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* The special case $m = 1$ of this formula is noteworthy for the particularly simple result ($q \leqq p'$) which it yields; while the special case $m = \frac{1}{2}$ gives us the result we should have obtained by reducing (1) to the binomial form by a change of dependent variable only.

If on the other hand we wished to consider, for this same equation (9), intervals lying at a great distance from the origin, none of the formulæ above yield good results, the best being again ($f$), which when $m = 1$ shows us that (9) is non-oscillatory in any interval of length $\pi/2$. A much better result may be obtained by letting

$$\varphi = \frac{1}{\sqrt{x}} \sin m(x - \alpha) \quad \left(\alpha < a < b < \alpha + \frac{\pi}{m}\right).$$

This function leads us to the inequality

$$1 \leqq m^2 - \frac{3m - 2}{4x^2},$$

which just fails to be satisfied for large values of $x$ when $m = 1$. We thus see that if $l$ is any positive constant less than $\pi$, a positive constant $M$ exists such that, in every interval of length $l$ throughout which $x > M$, the equation (9) is non-oscillatory. That the function we have just used should give us an interval which is nearly twice as long as that given by formula ($f$) is the more remarkable because when $x$ is large these two functions $\varphi$ are, throughout an interval of length $\pi$, very nearly proportional to each other. If they were exactly proportional they would obviously lead to the same result.

We now leave these illustrative applications to equation (9).

Although we originally deduced condition (8) and its special cases as an extension, to the general equation (1), of the condition $q \leqq 0$ which we had established for the binomial equation (2), it turns out that some of these conditions give results for equation (2) which go beyond the result from which we started. Thus if we apply ($f$) to the special case $p = 0$ we get the important theorem

*If throughout an interval of length less than $l$*

$$q \leqq \frac{\pi^2}{l^2},$$

*the equation* (2) *is non-oscillatory in this interval.**

Up to this point we have obtained merely sufficient conditions that a differential equation should be non-oscillatory.

* Cf. the proof here given of this familiar theorem of Sturm with that given by Picard (l. c.).

A condition which turns out to be also necessary is obtained by taking as the function $\varphi$ a solution of (1):

*A necessary and sufficient condition that* (1) *should be non-oscillatory in the interval* $a \leqq x \leqq b$ *is that it should have a solution which does not vanish in this interval.*

That this is a sufficient condition* is seen at once from the fact that either this solution or its negative satisfies conditions (7) and (8). To prove that it is also a necessary condition, assume that the interval $a \leqq x \leqq b$ is non-oscillatory, and consider the two solutions $y_1$ and $y_2$ of (1) which satisfy the conditions

$$y_1(a) = 0, \quad y_1'(a) > 0\,; \quad y_2(b) = 0, \quad y_2'(b) < 0.$$

Since neither of these solutions can vanish again in the interval, and since they are positive in the neighborhood of $a$ and $b$ respectively, they must be positive throughout the remainder of the interval. Accordingly $y_1 + y_2$ is a solution of (1) which is positive throughout the whole interval.

From the theorem just proved follows immediately this result:

*A necessary and sufficient condition that the equation* (*1*) *is non-oscillatory in the interval* $a \leqq x \leqq b$ *is that the solution of* (*1*) *which vanishes at* $a$ (*or, if we prefer, at* $b$), *but is not identically zero, does not vanish again in the interval.*

This theorem gives us what is theoretically a perfect test; we have merely actually to compute the solution of (1) which vanishes at $a$, in the form of a series say, and to see whether or not this solution vanishes again in the interval. The difficulties involved in the computation may of course be so great in any special case as to make this method practically useless.

This last theorem if applied to (9) shows us that this equation is non-oscillatory in the interval $a \leqq x \leqq b$, where $a$ is any small positive quantity and $b$ is the smallest positive root of the Bessel function $J_0(x)$, viz. $b = 2.40 \cdots$.†

The condition (8) can also be stated in the following form which again gives us a necessary as well as a sufficient condition:

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* This also follows at once from the well known theorem of Sturm: *Between two successive roots of a solution of* (1) *lies one and only one root of any linearly independent solution;* and conversely this theorem follows from the theorem of the text.

† This is actually the largest value that can be given to $b$, since every other solution of (9) has a root smaller than this. The proof of this fact is complicated by the presence of a singular point of (9) at $x = 0$. See, however, BULLETIN, March, 1897, p. 211.

*If $r$ is a real function of $x$ which, throughout the interval $a \leqq x \leqq b$, is continuous, and satisfies the condition $r \leqq 0$, then a necessary and sufficient condition that (1) be non-oscillatory in this interval is that the non-homogeneous equation*

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = r \tag{10}$$

*have a solution $y$ satisfying the relation*

$$y > 0 \quad (a \leqq x \leqq b).*$$

That this is really a necessary condition is seen from the fact that if (1) is non-oscillatory it has a solution positive throughout the interval, and by adding a sufficiently large positive multiple of this solution of (1) to an arbitrarily chosen solution of (10) we get the positive solution of (10) desired.

Another very important form into which condition (8) can be thrown is the following:

*The function $q_1$ being continuous in the interval $a \leqq x \leqq b$ and satisfying the condition*

$$q_1 \geqq q \qquad (a \leqq x \leqq b),$$

*the equation (1) will be non-oscillatory in this interval if the equation*

$$y'' + py' + q_1 y = 0 \tag{11}$$

*is non-oscillatory there.*

For if (11) is non-oscillatory it has a solution $\varphi$ positive throughout the interval $a \leqq x \leqq b$. Substituting $\varphi$ in the first member of (1) gives us, when we take account of the fact that $\varphi$ satisfies (11),

$$\varphi'' + p\varphi' + q\varphi = (q - q_1)\varphi \leqq 0 \qquad (a \leqq x \leqq b).$$

Thus $\varphi$ satisfies conditions (7) and (8), and therefore (1) is non-oscillatory.†

In conclusion I will mention that condition (7) may be replaced by the somewhat less restrictive condition

$$\varphi > 0 \qquad (a < x < b), \tag{7'}$$

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* This theorem may also be proved directly and the other theorems of this paper deduced from it.

† This theorem may be proved directly by means of the methods used by Sturm (*Liouville's Journal*, vol. 1, p. 106) and is in fact a special case of one of Sturm's theorems. From it may be deduced the other theorems of this paper.

*i. e.*, the case in which the function $\varphi$ vanishes at one or both ends of the interval need not be excluded. The interval on the $t$-axis would, however, then extend to infinity in one or both directions, and the fundamental theorem concerning equation (2) from which we started would on longer be sufficient, but would have to be replaced by a theorem which states that, if $q \leqq 0$, no solution of (2) which vanishes at a finite point can approach a finite limit as $x$ becomes either positively or negatively infinite, and that no solution of (2) can approach finite limits both when $x = +\infty$ and when $x = -\infty$.

The extension which our other theorems gain by the use of (7′) in place of (7) is easily seen. In using functions $\varphi$ which vanish at one of the ends of the interval it is useful to know that if $\varphi'$ also vanishes then $\varphi$ cannot possibly satisfy (8),—a fact whose proof we also omit.

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# CONCERNING REAL AND COMPLEX CONTINUOUS GROUPS.

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1. THIS paper aims to illustrate certain differences and certain analogies between related real and complex continuous groups. Lie's theory has been developed chiefly for the latter groups, the modifications necessary for real groups being treated quite briefly.

In §§ 2–4 are exhibited a real group in $m$ variables and a real group in $2m$ variables, each of $m^2$ parameters, such that the corresponding complex groups are of like structure. In §§ 5–8, it is shown for $m = 2$ that the two real groups have different structures. Of the three proofs given, the first two are analytic and involve little technical knowledge of group theory, while the third group is geometric and gives a better insight into the nature of the question.

In § 10, it is illustrated for the case $m = 2$ how the general $m$-ary linear homogeneous complex continuous group gives rise to an isomorphic $2m$-ary linear homogeneous real continuous group. Similarly, the complex projective groups lead to groups of birational quadratic transformations.