MUTH'S ELEMENTARTHEILER.

Theorie und Anwendung der Elementartheiler. Von Dr. P. MUTH. Leipzig, B. G. Teubner, 1899. xvi and 236 pages.

THE work under review covers an important subdivision of general invariant theory, a branch which deserves to be more widely known than it is. No doubt the cause of this neglect has been the lack of a text book, and Dr. Muth's monograph will help to remedy the defect.

The original problem which led to the series of investigations in this theory was that of the canonical reduction of two quadratic forms. In 1829 Cauchy published a paper on the secular inequalities of the planets,* in the course of which he showed that two quadratic forms can in general be reduced to sums of squares of the same variables. He also proved that the latent roots of the family† are all real, in the special case when one of the quadratic forms is positive for all real (non-zero) values of the variable, *i. e.*, is *definite.* Cauchy's results, though not perfectly general, cover most of the cases which occur in the first stages of geometry and dynamics. Jacobi (1834) found similar results by a somewhat different process.‡ Both Jacobi and Cauchy exclude the possibility of equal latent roots appearing in the problem.

The first systematic account of all possible types of two quadratic forms (allowing for equal latent roots) is to be found in Sylvester's paper (1851) on the contact of lines and surfaces of the second order.§ Here we meet with the idea of classification by means of *invariant factors.*|| Sylvester obtains 4 types of contact for conics, and 12 for quadrics; of course the algebraical possibility that the two forms differ only by a constant factor is trivial in a geometrical

* Exercices de mathématiques, vol. 4, p. 140 = Oeuvres (2d series), vol. 9, p. 174. Cauchy really discusses only the case when one of the original forms is already a sum of squares.

† If A, B are two given quadratic forms, the system of forms $uA + vB$ (u, v arbitrary parameters) will be called a *family* (as an equivalent for Kronecker's term *Schaar*). The *latent roots* are the values of the ratio $(-u : v)$ for which the determinant of the family vanishes; this determinant will be denoted by $|uA + vB|$.

‡ *Crelle*, vol. 12, p. 1 = Ges. Werke, vol. 3, p. 191.

§ *Phil. Magazine* (4th series), vol. 1, p. 119.

|| Weierstrass's *Elementartheiler*, and Sauvage's *élémentaire diviseur.* A definition is given in a later footnote on Weierstrass's work.

sense.* Unfortunately Sylvester did not prove that the invariant factors do constitute a *complete* set of invariants for two quadratic forms; nor did he explain how to effect the reduction of given forms to his standard types. †

In 1855 Cayley pointed out that Sylvester's ideas could be applied to the classification of homographies (*i. e.*, one-to-one transformations of space, or collineations). ‡ Here will be found a general formula for the number of possible collineations in space of any dimension n; and the calculation is given for $n = 1, 2, \cdots, 11$. Cayley gives symbols to represent each type by means of the indices of its invariant factors and enumerates the possibilities for $n = 1, 2, 3, 4$.§ In concluding the paper he says: "*Je reviendrai à cette théorie à une autre occasion*;" but apparently this promise was not fulfilled.

The next advance in the general theory was made in 1858 by Weierstrass, who gave a general method of reducing two quadratic forms to sums of the same squares.‖ He proved that if one of the forms be definite, the reduction is still possible, even if equalities exist among the latent roots. Using this result in connection with the dynamical problem of small oscillations about a position of equilibrium, Weierstrass showed that the stability is not destroyed by the presence of equal periods in the system; both Lagrange and Laplace supposed that equal periods would involve instability.¶ Weierstrass's theorem is commonly attributed

* In the algebraical problem we must further allow for the case of no contact, so that in all we have 6 and 14 as the numbers of the distinct types of families of quadratic forms ($n = 3$ or 4).

† To illustrate the simplification introduced by the use of invariant factors in geometry, the reader may compare the condition for double contact of two conics as found by Salmon (Conics, p. 346) with Sylvester's; which is simply that there should be two equal *linear* invariant factors of the determinant of the family.

‡ "Recherches sur les matrices dont les termes sont des fonctions linéaires d'une seule indéterminée." *Crelle*, vol. 50, p. 313 = Coll. Math. Papers, vol. 2, p. 216.

§ In his list for $n = 4$ the two symbols $\begin{matrix}31\\1\end{matrix}$, $\begin{matrix}21\\21\end{matrix}$ are omitted: Muth would write these [(21)1], [(11)(11)]. This is obviously an oversight on Cayley's part, as he states that 14 types are possible, but gives only 12. The indices of the invariant factors are the differences between Cayley's numbers.

‖ *Berl. Monatsberichte*, 1858, p. 207 = Ges. Werke, vol. 1, p. 233.

¶ The point of the theorem will probably be grasped better by considering a special (imaginary) example. Suppose we had a system with kinetic and potential energies, $T = x'y'$, $V = p^2xy + \frac{1}{2}x^2$, then the equations of motion are $x'' = -p^2x$, $y'' = -(p^2y + x)$, and corresponding to a term $A \sin pt$ in x we have one $\frac{At}{2p} \cos pt$ in y. Weierstrass proves that such a case is impossible in a real dynamical system.

by English writers* to Routh who rediscovered the theorem in his Adams prize essay (1877). Routh's elementary proof of the theorem requires to be supplemented by Weierstrass's algebraical theorem.

A beautiful and elementary proof of Weierstrass's theorem was given by Kronecker† in 1868, which appears to be less familiar than it ought to be.

Weierstrass‡ finally completed the theory of quadratic forms in 1868; and extended his theorems to the more general idea of bilinear forms. He proved that two families of forms are capable of transformation into each other if (and only if) the invariant factors§ of the two families are the same. The only case of exception to Weierstrass's results occurs when the determinants of the two families are identically zero; and this case was examined by Kronecker in a paper immediately following Weierstrass's (and quoted above). The complete statement of the conditions of equivalence in every case has been effected by Kronecker in a number of memoirs.||

Invariant factors presented themselves from another point of view to H. J. S. Smith, who encountered them in connection with linear equations with integral coefficients.¶ He proves that many of the results known for the invariant factors of a family of forms hold for those of a matrix of integers.

We turn next to the historical development of Cayley's

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* *E. g.*, Thomson and Tait, Natural Philosophy, vol. 1, § 343 *e* and *m*; where the reader will find some useful remarks on the physical side of the theorem. Routh was led to his theorem by the consideration of linear differential equations with constant coefficients; and in this connection obtained some other results due to Weierstrass, which were given originally in a communication to the Berlin Academy (in 1875), but were only published in vol. 2 of his works.

† *Berl. Monatsberichte*, 1868, p. 339 = Ges. Werke, vol. 1, p. 165.

‡ *Berl. Monatsberichte*, 1868, p. 310 = Ges. Werke, vol. 2, p. 19.

§ *Definition:* Suppose that the determinant of the family $(uA + vB)$ has a repeated factor $(au + bv)^p$; and further that *every* first minor is divisible by $(au + bv)^q$ but not by $(au + bv)^{q+1}$; *every* second minor by $(au + bv)^r$; and so on. Then the numbers $p - q$, $q - r$, ⋯ are the indices of the invariant factors to the base $(au + bv)$; they have the property $(p - q) \overline{\gtrless} (q - v) \overline{\gtrless} \cdots \overline{\gtrless} 1$, which was recognized by Cayley (1 c., supra). The invariant factors can also be defined rationally, by means of highest common divisors (a remark due to Smith and Kronecker).

|| For the results (which are rather long) the reader may consult § 8 of Dr. Muth's book; there is another investigation and a list of papers in the *Proc. Lond. Math. Soc.*, vol 32 (1900), p. 98.

¶ Math. Papers, vol. 1, p 367, and vol. 2, p. 623; some of Smith's results were published before the corresponding theorems for a family. For other references in this direction consult the Encyklopädie der Math. Wiss., vol. 1, pp. 582–597.

theory of matrices*; the foundations were given by him in 1855† and his first long memoir on the subject appeared in 1858‡; together with an application (in a consecutive paper) to Hermite's theory of the automorphic substitutions of a given quadratic form. Some years later (1867) Laguerre§ published an independent account of his calculus of "linear systems" which are virtually Cayley's matrices. But in 1853, Hamilton|| had given certain results relating to a "linear and vector" function; which is essentially a matrix of order 3. For instance he proved that such a function satisfies a certain cubic equation with constant coefficients, and he found the reciprocal function. He therefore anticipated Cayley's theorem on matrices¶ (for the special case $n = 3$). Further, in 1867, he showed (Elements of quaternions) that the cubic may reduce to a quadratic (a special case of Frobenius's theorem referred to below). Probably this is the theory alluded to by Study (in the reference quoted above).

In a letter to *Nature* (vol. 44 (1891), p. 79), Professor J. W. Gibbs claims for Grassmann the first suggestion of the theory of matrices (Ausdehnungslehre, 1st ed., 1844). In this connection Whitehead (Universal algebra, Cambridge, 1898, vol. 1, p. 248) refers only to the second edition (1861) of Grassmann's work and implies that Cayley had anticipated Grassmann. I am unable to give any opinion on this point; nor do I know the extent of Grassmann's results.

In 1878 Frobenius** pointed out the important connection between bilinear forms and matrices (or linear substitutions). Being familiar with the results of Weierstrass and Kronecker on the equivalence of families of bilinear forms,†† he was naturally led to introduce the idea of invariant factors into the theory of matrices. This step has proved most fruitful in both theories; it enabled Frobenius to give the first general proof of Cayley's theorem, and to modify the theorem in the case when some of the latent roots of the matrix are

* Study in his report on complex units (Encyklopädie der Math. Wiss., vol. 1, p. 169) states that a theory had been hinted at by Hamilton. (See below.)

† *Crelle*, vol. 50, p. 282 = Coll. Math Papers, vol. 2, p. 185.

‡ *Lond. Phil. Trans.*, vol. 148, p. 17 = Coll. Math. Papers, vol. 2, p. 475.

§ *Jour. École Polyt.*, vol. 25, p. 215.

|| Lectures on quaternions, pp. 559–569.

¶ That is: A matrix of order n satisfies an identical equation of order n. Apparently Cayley himself only verified the theorem up to $n = 3$; his proof is for the case $n = 2$ only.

** *Crelle*, vol. 84, p. 1.

†† It may be remarked that Frobenius gives a very convenient summary of these results in §6 of his paper.

equal.* Amongst other methods suggested in this paper is the use of the expansion of $(\lambda E - A)^{-1}$ to investigate properties of A, or rather of the latent roots of A; the expansion is made in powers of $(\lambda - a)$ (a being any latent root), and starts with a term in $(\lambda - a)^{-\alpha}$, $(\lambda - a)^{\alpha}$ being the first invariant factor of $|\lambda E - A|$, which belongs to a.†

Among other applications of this method we may mention Frobenius's definition ‡ of *any* function of A: provided that $f(A)$ can have a meaning, it is equal to the sum of the residues of $f(\lambda)\ (\lambda E - A)^{-1}$ taken for all the latent roots of A. While speaking of this, I should call attention to what appears to be an oversight in the Clark University decennial volume (Worcester, Mass., 1899); in the report on the mathematical faculty, it is stated that Sylvester (in 1882) was the first to give an expression for the general power of a matrix. But according to Frobenius, Stickelberger (in 1881) gave the expression, and moreover for the case of repeated latent roots; while Sylvester assumes all the latent roots to be unequal.§ For the generalized function of a matrix Sylvester was the first to give a result ‖ (but with the same restriction on the latent roots); it is claimed in the Clark volume that the extension to the case of equal latent roots was made by Professor Henry Taber (in 1893,

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* By "latent roots" of a matrix A, we mean the roots of the determinantal equation $|\lambda E - A| = 0$, where E is the unit matrix. Probably the simplest proofs of the theorems are those of Frobenius (*Berl. Sitzungsberichte*, 1896, p. 604), though the same proof of Cayley's theorem had been given by Buchheim (*Mess. of Math.*, vol. 13 (1883), p. 62).

† To illustrate the method I collect six theorems on special matrices giving their authors and dates of publication. All the theorems can be proved most simply by Frobenius's method. (*L. R.* denotes latent roots; *I. F*, invariant factors.)

A symmetrical	*L. R.* real (Cauchy, 1829). *I. F.* linear (Weierstrass, 1850).
A orthogonal	*L. R.* $e^{i\phi}$ (Brioschi, 1854). *I. F.* linear (Frobenius, 1878).
A alternate	*L. R.* imaginary (Weierstrass, 1879). *I. F.* linear (Weierstrass, 1879).

All these theorems were rediscovered by Professor Henry Taber (*Proc. Lond. Math. Soc.*, vol. 22 (1891), p. 449).

‡ *Berl. Sitzungsberichte*, 1896, p. 7.

§ After defining $f(A)$ as above, Frobenius (l. c., p. 11) says—"In dieser Weise hat Stickelberger in seiner akademischen Antrittsschrift: *Zur Th. d. lin. Diffgl.* (Leipzig, 1881) die allgemeine Potenz definirt und . . . benutzt. Eine weniger genaue Definition giebt Sylvester: *Sur les puissances et les racines des subst. lin.* (*Comptes rendus*, vol. 94 (1882), p. 55)." Professor Stickelberger informs me (in a private letter) that some copies of his paper were published at the end of 1880.

‖ *Johns Hopkins University Circulars*, vol. 3 (1882); of course the step from the generalized power to the general function is almost self-evident.

1894). But Buchheim* (in 1886) had already made the extension, and in a more concise form; both forms are, of course, contained in Frobenius's statement quoted above.

In referring to Frobenius we should remark that he has a claim to be a joint discoverer of a theorem, frequently attributed to Jordan,† to the effect that every *periodic* linear substitution can be reduced to the form

$$y_r' = e_r y_r$$

where e_r is a root of unity. This theorem has been recently taken up afresh by E. H. Moore and H. Maschke.‡ On the whole, the investigation of Frobenius seems the most direct and the clearest of all that I have seen; though, as it requires an elementary acquaintance with the theory of matrices, students of group-theory and differential equations may prefer an independent proof.

Of recent developments, we may refer to a recent paper by S. Kantor§ (1900) in which the theory of invariant factors of higher kinds (Stufen) is examined by the aid of geometry in space of n^2 dimensions. This appears to be an entirely fresh departure.

Dr. Muth's book is the first published account of the theories which we have sketched above;‖ our thanks are due to the author for having collected so many useful results into one convenient volume. On the other hand it may be questioned if readers beginning the subject will not do well to modify his arrangement. Thus §1, on general

* *Phil. Magazine*, 5th series, vol. 22, p. 173.

† On p. 16 of his paper, Frobenius shows that for periodic substitutions all the invariant-factors are linear and that the latent roots are roots of unity. On p. 21 it is stated that the reduced forms depend only on the invariant factors (using Weierstrass's results for bilinears), and the combination of these two facts leads directly to the theorem as given. Jordan's statement occurs later in the same volume of *Crelle* (p. 112); there can be little doubt as to their independence.

‡ *Math. Annalen*, vol. 50 (1898), pp. 215 and 220; cf. L. E. Dickson, "Report on progress in the theory of linear groups," BULLETIN, vol. 6 (1900), p. 13.

§ *Monatshefte für Math. und Phys.*, vol. 11, p 193; another way of applying n dimensional geometry to matrices has been given by Buchheim (*Proc. Lond. Math. Soc.*, vol. 16 (1885), p. 63) after Grassmann. We may consult also Whitehead's Universal algebra (vol. 1, p. 248).

‖ Two fairly complete accounts of the theory have appeared recently: Ed. Weyr, *Monatshefte für Math. und Phys.*, vol. 1 (1890), p. 163; Sauvage, *Ann. Ecole Norm. Sup.* (3d series), vol. 8 (1891), p. 285, and vol. 10 (1893), p. 9. Neither of these covers so much ground as Dr. Muth's work For sketches and references we may consult also F. Meyer's reports on general invariant theory; *Jahresbericht der Deutschen Math. Ver.*, vol. 1 (1890), p. 106 and Encyklopädie der Math. Wiss., vol. 1, p. 327.

properties of invariant factors, will probably be found one of the hardest in the book. In §2 will be found a good account of the theory of "multiplying" bilinear forms; the author follows Frobenius in preference to Cayley,* and introduces a matrix only as a picture (Bild) of the bilinear form. The theory of forms (or systems) with elements which are integers in various regions of rationality, occupies §§3–5 and 18. It may be pointed out that these methods can be applied† to deduce the principal theorems of §1.

In §§6–9 the theory of equivalence of families of forms is considered by the methods of Weierstrass and Kronecker; it seems to me that a somewhat easier introduction to the theory will be found in Darboux's paper on quadratic forms;‡ and in Stickelberger's extension of the same to bilinears;§ after which Muth's work will follow. It may be questioned if any advantage is obtained by the use of double suffixes in the reduced forms, at any rate in the tables (pp. 91, 116, 124, 133). The geometrical interpretation of the results on p. 124 by means of conics should be noted; this will be found in Sylvester's paper, already noted.

The remainder of the book is occupied chiefly with special applications; §10 contains Frobenius's method of reducing a given bilinear form by means of congruent substitutions;‖ § 11 gives a method of reducing a linear substitution to Jordan's canonical form, though it seems to me that the most practical method for reducing any substitution (whose latent roots are known) is that due to Jordan himself.¶

An application of the results of §11 is made in §16 to the theory of a system of linear differential equations with constant coefficients; this seems to be due to Weierstrass.

In §§12, 13, we have an account of linear substitutions which are automorphic for a given form; but it ought to be remarked that the first result in this direction is due to

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* It is somewhat remarkable that Dr. Muth nowhere refers to Cayley—not even in his historical account of the subject.

† Cf. Hensel, *Crelle*, vol. 114 (1894), pp. 25 and 109. Two papers related to this point of view have just been published by Dr. Muth himself, *Crelle*, vol. 122 (1900), pp. 84 and 89.

‡ *Liouville's Jour. de Math.*, 2d series, vol. 19 (1874), p. 347.

§ *Crelle*, vol. 86 (1878), p. 20.

‖ This problem was attacked by Kronecker (in 1866) in connection with Weierstrass's general theta functions. Kronecker finally settled the problem in 1874, after some controversy with Jordan. The problem is of interest in certain dynamical questions, as well as in connection with theta functions.

¶ Cours d'Analyse, vol. 3, Art. 143; a reproduction is given in Craig's book on Differential equations. A series of papers on this subject will be found in vols. 30-32 of the *Lond. Math. Soc. Proc.* (1899–1900).

Hermite,* whose conclusions were translated into the language of matrices by Cayley. No references are given to recent researches on the so-called "singular" cases of automorphic substitutions.† It is perhaps worth while to call attention to a result of Loewy's ‡ which shows that, by using a Hermite form (with complex coefficients) in place of the real symmetrical and alternate forms hitherto considered, we avoid all consideration of the singular cases. Loewy's paper contains many other valuable results and will repay careful study.

In §§14, 15, Dr. Muth obtains certain results on linear invariant factors; here we should note Klein's results in his inaugural dissertation (1868)§, which have been further extended by Loewy.|| §17 contains an elaborate account of all possible collineations in space of n dimensions, which is afterwards applied to spaces of two and three dimensions. As stated before, the first attempt to classify such transformations is due to Cayley, who did not, however, include the possibility of "singular" collineations; as an illustration of what is meant by such collineations we may take the perspective of ordinary drawing, in which a point of the picture represents all the points on a certain line through the eye. Another form of classification has been given by Whitehead.¶

Apparently it is usual to conclude a review with a list of misprints. Of those which I have noticed, not many need delay the reader. In several places the short vertical lines

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* *Crelle*, vol. 47 (1854), p. 309 ($n=3$) and *Camb. and Dublin Math. Jour.*, vol. 9 (1854), p. 63 ($n=4$). In connection with Muth's table on p. 172 we may refer to Jordan, *Liouville's Jour. de Math.* (4th series), vol. 4 (1888), p. 349.

† See § 11 of Frobenius's paper; it had been considered previously (in the special case $n=3$) by Bachmann, Tannery and Hermite, without the aid of matrices. More recent investigations are those of Loewy in *Math. Annalen*, vol. 48 (1897), p. 97 and vol. 49, p. 448; of Taber, *Proc. Lond. Math. Soc.*, vol. 24 (1893), p. 290, and vol. 26 (1895), p. 364; also in *Math. Annalen*, vol. 46 (1895), p. 561, and a number of papers in the BULLETIN. In connection with Taber's work we should notice a paper by Rettger (*Amer. Jour. of Math.*, vol. 22 (1900), p. 62, which gives similar results from the point of view of general continuous group theory. (See also the Clark volume, already quoted.)

‡ *Nova Acta Leopoldina*, vol. 71 (1898), p. 379 = *Math. Annalen*, vol. 50, p. 569. Cf. *Göttinger Nachrichten*, 1900, p. 298.

§ Reprinted in *Math. Annalen*, vol. 23; the theorem alluded to is given on pp. 561, 562 and is, from the point of view of Klein's dissertation, a subsidiary result. In the reprint Klein remarks that the theorem seems not to have been sufficiently considered.

|| *Crelle*, vol. 122 (1900), p. 53.

¶ Universal algebra, vol. 1, p. 316.