THE OCTOBER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

A REGULAR meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York City on Saturday, October 27, 1900, extending through the usual morning and afternoon sessions. Thirty-two persons were in attendance, including the following twenty-five members of the Society:

Professor E. W. Brown, Professor F. N. Cole, Dr. J. W. Davis, Dr. W. S. Dennett, Professor T. S. Fiske, Mr. A. S. Gale, Miss Ida Griffiths, Miss Carrie Hammerslough, Dr. A. A. Himowich, Professor Harold Jacoby, Mr. S. A. Joffe, Dr. Edward Kasner, Mr. C. J. Keyser, Dr. G. H. Ling, Dr. Emory McClintock, Dr. G. A. Miller, Professor R. W. Prentiss, Dr. P. L. Saurel, Professor P. F. Smith, Professor J. H. Van Amringe, Professor E. B. Van Vleck, Professor J. M. Van Vleck, Professor L. A. Wait, Miss E. C. Williams, Professor R. S. Woodward.

Professor Thomas S. Fiske, Vice-President of the Society, occupied the chair. The Council announced the election of the following persons to membership in the Society: Professor G. L. Brown, South Dakota Agricultural College, Brookings, So. Dak.; Mr. C. H. Davis, New York, N. Y.; Dr. D. N. Lehmer, University of California, Berkeley, Cal.; Miss Ida M. Schottenfels, Chicago, Ill.; Professor F. D. Sherman, Columbia University, New York, N. Y.; Mr. Burke Smith, Northwestern University, Evanston, Ill. Twenty-five applications for membership were received.

It was decided to hold the next summer meeting and colloquium of the Society at Cornell University. A committee, consisting of Professors W. F. Osgood, James Pierpont, J. H. Tanner, and H. S. White, was appointed to coöperate with the Secretary in making the necessary arrangements. The conditions are favorable for extending the colloquium through a period of two or more weeks.

The following papers were read at this meeting:

(1) Professor MAXIME BÔCHER: "On linear dependence of functions of one variable."

(2) Professor DAVID HILBERT: "Ueber Flächen von constanter Gauss'cher Krümmung."

(3) Professor E. O. LOVETT: "Three notes on the geometry of contact transformations."

(4) Dr. G. A. MILLER: "On a theorem in substitutions."

(5) Mr. H. W. KUHN: "Several theorems on imprimitive groups."

(6) Professor S. L. PENFIELD ; "The solution of spherical triangles by graphical methods, with exhibition of scales and protractors."

(7) Miss I. M. SCHOTTENFELS : "On a set of definitional functional properties for the analytical function

$$f(z) = \frac{\tan \pi z}{\pi}."$$

(8) Professor P. F. SMITH : "Geometry within a linear spherical complex."

(9) Dr. E. J. WILCZYNSKI : "Invariants of systems of linear differential equations."

Professor Hilbert's paper was presented to the Society through Professor E. H. Moore, Mr. Kuhn's through Dr. G. A. Miller, and Miss Schottenfels's through the Secretary. Professor Penfield was introduced by Professor P. F. Smith. In the absence of the authors, Mr. Kuhn's paper was read by Dr. Miller, and the papers of Professor Bôcher, Professor Hilbert, Professor Lovett, Miss Schottenfels, and Dr. Wilczynski were read by title.

Professor Bôcher's paper appears in the present number of the BULLETIN. Abstracts of the remaining papers are given below.

Professor Hilbert's paper will appear in an early number of the *Transactions*. Following Beltrami, it is known that a piece free of singularities of a surface of negative constant curvature makes real in the domain of ordinary euclidean space a piece of a Lobachevsky (non-euclidean) plane, the straight lines of the Lobachevsky plane corresponding to the geodetic lines of the surface of negative constant curvature and the distances and angles in the plane being the real distances and angles on the surface. Of the known surfaces of negative constant curvature there is none which in all its extent is continuous and has a continuously varying tangent plane ; on the contrary all such known surfaces have singular lines beyond which a continuation of the surface with preservation of the properties just mentioned is impossible. The first part of this memoir concludes with the theorem that there exists no surface of negative constant curvature which in all its extent is continuous and has a continuously varying tangent plane, and hence that the complete Lobachevsky plane cannot by the method of Beltrami be realized in ordinary euclidean space.

The principal theorem of the second part is this: On any simlpy or multiply connected closed region entirely in the finite and free of singularities lying on an analytic surface of positive constant curvature + 1 the greater of the two principal radii of curvature of the surface at the various points of the region takes its maximum value at a point of a boundary, and at an interior point only if the surface is a part of the sphere of radius 1. From this follows immediately a new proof of the theorem proved by Liebmann (*Göttinger Nachrichten*, 1899) that the sphere of radius 1 is the only closed analytic surface free of singularities and of positive constant curvature +1.

Professor Lovett's paper consists of three notes. The first is a collection of original problems relative to the contact transformations between the principle elements of ordinary space, namely points, planes, straight lines, and spheres, and has been sent to the *Annals of Mathematics* for publication.

The second constructs the invariants of an n dimensional space as was done for four dimensional space in a recent number of the BULLETIN. It has been offered to the *American Journal of Mathematics*.

The third determines all contact transformations between singular manifoldnesses of surface elements and employs the results to effect the integration of the equations defining certain other categories of transformations. A surface element is the ensemble of a point and a plane through it, the dimensions of the plane being infinitesimal. Two surface elements are said to be associated when the point of each lies in the plane of the other to terms of the first order. An element manifoldness of surface elements is an aggregate such that every element of the aggregate is associated with its infinitely near neighboring element. These notions of surface element and element manifoldness were introduced by Lie. Now among these element manifoldnesses there exist certain ones which enjoy the property that every element of the aggregate is associated not only with its nearest neighbor but also with every other element of the family. Such element manifoldnesses might be called singular. There are but three varieties of such manifoldnesses, namely, the aggregates of all the surface elements of a point, of all those appertaining to a straight line, and of all the elements belonging to a plane. If it be proposed then to determine all the contact transformations which change singular element manifoldnesses into such, the following cases call for further examination: 1° those which change point into point; 2° those which change point into point and plane into plane; 3° those which change point into plane and plane into point; 4° those which change straight line into straight line; 5° those which change plane into plane. The first three categories are well known to consist respectively of the infinite group of all extended point transformations, of the finite group of all projective transformations, and of all dualistic transformations. The fourth category is equally well known to consist of the second and third categories, a result which yields itself immediately from the above definition of singular element manifoldness. The fifth category which does not seem to have been constructed is readily designed geometrically from the products DPD, where D is the general dualistic transformation and P is an arbitrary point transformation.

Dr. Miller's paper is in abstract as follows: Let s_1, s_2 be any two substitutions which involve the same n letters, and let G_1, G_2 be the groups which are composed of all the substitutions in these n letters that are commutative with s_1, s_2 respectively. The necessary and sufficient condition that G_1, G_2 represent the same substitution group is that s_1, s_2 are similar. All the substitutions of degree m that are commutative with s_1 form a group. When $m = n + b$ and this group involves all the letters of s_1, it is the direct product of G_1 and the symmetric group of degree b. Two dissimilar substitutions of degree n give rise to two distinct groups of degree m. Hence every substitution s_1 gives rise to an infinite number of substitution groups, each of which is distinct from the infinite number of groups that are obtained by the same process from any substitution that is not similar to s_1.

Mr. Kuhn's paper relates to imprimitive groups that contain more than one set of systems of imprimitivity. Let G denote any regular group. It is proved that the number of sets of systems of imprimitivity that can be found for G is equal to the number of its subgroups (not including identity). The cyclical group of order p^2 is therefore the only regular group that contains just one set of systems of imprimitivity. Any set of systems that corresponds to a subgroup that contains no invariant subgroup of G besides identity is permuted by the substitutions of G according to a transitive group that is simply isomorphic to G; any other set is permuted according to a transitive group that has a $(1, \alpha)$ isomorphism to G, α being greater than unity.

Let G' denote any imprimitive group that has more than one set of systems of imprimitivity. When the substitutions of G' permute all its sets of systems according to primitive groups and when none of the heads is identity, then

(1) No two of the heads of G' can contain a common substitution besides identity, and hence

(2) Each substitution of any other given head is commutative to each substitution of any other head.

(3) Each head contains at least one substitution whose degree equals the degree of G', and

(4) Each head is formed by establishing a simple isomorphism among its transitive constituents.

After a brief statement of the principles of the stereographic projection Professor Penfield exhibited a number of devices for plotting. One of these consisted of sheets upon which were printed a graduated circle of 14 cm. diameter and four scales, as follows: No. 1, giving the radii of stereographically projected arcs of great circles. No. 2, giving the radii of stereographically projected arcs of vertical small circles. No. 3, giving the degrees of a vertical circle stereographically projected upon a diameter. No. 4, giving the radius of the circle divided into one hundred parts. Other devices consisted of two protractors, one having along its base line a scale giving the degrees of a circle stereographically projected, the other used in measuring and for that purpose printed on transparent celluloid, giving the arcs of vertical small circles stereographically projected. There were also exhibited devices of a similar nature, but made up on a large scale for blackboard demonstrations.

Right and oblique angled triangles were solved graphically on the blackboard, and the graphical solution of similar problems carefully executed on the printed sheets were exhibited, indicating that within a circle of 14 cm. diameter computations could be made with an average error not exceeding seven minutes.

There were also exhibited a patented arm protractor for drawing and measuring plane angles, and goniometers especially devised for measuring crystals.

Special emphasis was laid upon the importance of graphical methods for making the studies of plane and spherical trigonometry and geometry more real to students, as a means of checking numerical calculations, and as a rapid method for obtaining approximate results.

The subject of Miss Schottenfels's paper was suggested by the investigation of Professor E. H. Moore for the function $\frac{\sin \pi z}{\pi}$ in the *Annals of Mathematics*, volume 9, January, 1895. A set of definitional functional properties are obtained for the function $f(z) = \frac{\tan \pi z}{\pi}$, and it is shown that there is one and only one function $f(z)$ possessing these properties. The external exponential factor is then determined in the expression of the function as a Weierstrassian infinite product of primary factors.

Professor Smith's paper takes up the relation of the geometry of reciprocal radii to the geometry within a spherical complex, as determined by the single isomorphism of the corresponding groups, viz., these groups are transformed into each other by contact transformation, whose properties were defined and studied by the author in the paper published in the *Transactions*, October, 1900, "On surfaces enveloped by spheres belonging to a linear spherical complex."

In a paper, soon to be published in the *American Journal of Mathematics*, Dr. Wilczynski has shown that the most general point transformation which converts a general system of n linear homogeneous differential equations into another of the same form and order is

$$(1) \qquad x = f(\xi), \quad y_k = \sum_{i=1}^{n} a_{ki}(\xi)\eta_i, \qquad (k = 1, 2, \cdots, n),$$

where $f(\xi)$ and $a_{ki}(\xi)$ are arbitrary functions of ξ.

Functions of the coefficients and of their derivatives which have the same value for the original as for the transformed system are called invariants. Covariants are such invariant functions involving also $y_1, \cdots, y_n, y_1', \cdots, y_n'$, etc.

Equations (1) define an infinite group G in the $n+1$ variables $x, y_1, \cdots, y_n$. A subgroup of this is that for which x remains unchanged and only $y_1, \cdots, y_n$ are transformed. Functions of the coefficients invariant under this subgroup are called *seminvariants*. These can be found by setting up, by the method of infinitesimal transformations, the partial differential equations which they satisfy. The invariants are functions of the seminvariants.

The actual computation of the seminvariants and invariants is carried out only for a system of two equations each of the second order, *i. e.*, for the system

$$(2)\qquad y_i'' + p_{i1}y_1' + p_{i2}y_2' + q_{i1}y_1 + q_{i2}y_2 = 0, \qquad (i=1, 2),$$

but the methods employed admit of generalization.

If we put

$$(3)\qquad \begin{aligned} u_{11} &= 2p_{11}' - 4q_{11} + p_{11}^2 + p_{12}p_{21}, \\ u_{12} &= 2p_{12}' - 4q_{12} + p_{12}(p_{11} + p_{22}), \\ u_{21} &= 2p_{21}' - 4q_{21} + p_{21}(p_{11} + p_{22}), \\ u_{22} &= 2p_{22}' - 4q_{22} + p_{22}^2 + p_{12}p_{21}, \end{aligned}$$

then two seminvariants are

$$(4)\qquad I = u_{11} + u_{22}, \quad J = u_{11}u_{22} - u_{12}u_{21}.$$

Four quantities v_{ik} are now formed, cogredient with the quantities u_{ik}, viz.,

$$(5)\qquad \begin{aligned} v_{11} &= 2u_{11}' + p_{12}u_{21} - p_{21}u_{12}, \\ v_{12} &= 2u_{12}' + (p_{11} - p_{22})u_{12} - p_{12}(u_{11} - u_{22}), \\ v_{21} &= 2u_{21}' - (p_{11} - p_{22})u_{21} + p_{21}(u_{11} - u_{22}), \\ v_{22} &= 2u_{22}' - p_{12}u_{21} + p_{21}u_{12}, \end{aligned}$$

and a new independent seminvariant is

$$(6)\qquad K = v_{11}v_{22} - v_{12}v_{21}.$$

A third set of quantities w_{ik} is formed from v_{ik} and p_{ik}, in exactly the same way as v_{ik} from u_{ik} and p_{ik} in (5), i. e.,

$$(7)\qquad \begin{aligned} w_{11} &= 2v_{11}' + p_{12}v_{21} - p_{21}v_{12}, \\ &\text{etc., etc.;} \end{aligned}$$

and since the v_{ik}'s were cogredient with the u_{ik}'s, so are the w_{ik}'s. Then

$$(8)\qquad L = w_{11}w_{22} - w_{12}w_{21}$$

is a new seminvariant.

All seminvariants are functions of I, J, K, L, and their derivatives.

Some of the invariants are

$$\theta_4 = I^2 - 4J \quad ,$$

$$\theta_6 = 2I(I^2 - 4J) + 5\,(K - I'^2) + 4(K - 2J'' + II'').$$

etc., these being of weight 4 and 6 respectively.

If the system of equations (2) is equivalent to a single linear differential equation of the second order, all of the invariants vanish, and conversely.

Any system of form (2) can be transformed into another for which $p_{ik} = 0$. This is called the semicanonical form of the system. The subgroup G' of G which leaves this form unchanged is examined. But we can fulfill the further condition $q_{11} + q_{22} = 0$, and a system for which both $p_{ik} = 0$ and $q_{11} + q_{22} = 0$, is said to be in the canonical form. The subgroup G'' of G' which leaves this unchanged is a finite group of very simple form, and has some additional invariants of the form called quadri-derivatives by Forsyth.

Only a few simple results about covariants are mentioned. This and further generalizations are left for a future paper.

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ON LINEAR DEPENDENCE OF FUNCTIONS OF ONE VARIABLE.

BY PROFESSOR MAXIME BÔCHER.

It is ordinarily stated that the identical vanishing of the determinant

$$D = \begin{vmatrix} u_1 & u_2 & \cdots u_n \\ u_1' & u_2' & \cdots u_n' \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ u_1^{[n-1]} & u_2^{[n-1]} & \cdots u_n^{[n-1]} \end{vmatrix}$$

is a sufficient* condition for the linear dependence of the functions $u_1(x), u_2(x), \cdots, u_n(x)$. This is perfectly true if the u's are analytic functions of the complex variable x. *This condition is however no longer sufficient if we are dealing with functions of a real variable*, even though these functions possess derivatives of all orders for every real value of x. The truth of this statement will be seen from the example of two functions u_1 and u_2 defined as follows:

$$u_1 = \begin{cases} e^{-1/x^2} & (x \neq 0) \\ 0 & (x = 0) \end{cases} \qquad u_2 = \begin{cases} e^{-1/x^2} & (x > 0) \\ 0 & (x = 0) \\ 2e^{-1/x^2} & (x < 0) \end{cases}$$

* It is of course a necessary condition provided the u's have derivatives of the first $n - 1$ orders.