permanent bureau. Some questions are better left undecided. International agreement is not wanted on all points; international rivalry and emulation still have their part to play, helped by the international friendships that are promoted by such gatherings as these international congresses.

In conclusion, I must express my thanks to many of those whose names appear in this report for the assistance I have received from them in its preparation.

CHARLOTTE ANGAS SCOTT.

BRYN MAWR COLLEGE,
October, 1900.

THE FORTY-NINTH ANNUAL MEETING OF THE AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE.

THE forty-ninth annual meeting of the American Association for the Advancement of Science, which was held at Columbia University June 23–30, was from the point of view of scientific work one of the most successful in the history of the Association. Sixteen affiliated societies met with the Association and contributed greatly to the importance and interest of the meeting. Two of these, the American Mathematical Society and the Astronomical and Astrophysical Society, held joint sessions with Section A.

The next meeting of the Association will be held in Denver during the last week of August, 1901, under the presidency of Professor Minot of the Harvard Medical School. Professor James McMahon, Cornell University, will be vice-president of Section A, and Professor H. C. Lord, Ohio State University, will be secretary. Forty-one out of the total number of forty-nine annual meetings of the Association have been held during the month of August, while the recent New York meeting was the first that was held in June. The next meeting will be farther west than hitherto, but it seemed to be the general opinion that this was desirable in order to extend the influence of the Association. Pittsburg was recommended as the place of meeting in 1902.

The meetings of the section of mathematics and astronomy were well attended. The officers of this section were: vice-president, Asaph Hall, Jr.; secretary, W. M. Strong; councilor, Ormond Stone; sectional committee, Alexander MacFarlane, J. F. Hayford, C. L. Doolittle, W. M. Strong, Harold Jacoby, Edgar Frisby, W. S. Eichelberger. The Council elected the following mathematicians and astronomers to fellowship in the Association: E. S. Crawley, G. A. Hill, W. J. Humphreys, H. C. Lord, L. H. Orleman, Mary W. Whitney, P. S. Yendell.

The opening address of the Vice-President of Section A, "On the teaching of astronomy in the United States," has been published in *Science*, July 6, 1900, pp. 15–20. In the absence of the Vice-President, Simon Newcomb, C. L. Doolittle and Ormond Stone presided at the meetings of this Section. The following list of papers read before the Section does not include those of the American Mathematical Society and of the Astronomical Society which were read at the joint sessions of these societies and Section A.

(1) President H. S. PRITCHETT: "The functions, organization, and future work of the U. S. Coast and geodetic survey."

(2) Mr. J. F. HAYFORD: "The precise level net of the U. S. and a new levelling instrument."

(3) Professor T. J. J. SEE: "The propagation of the tide wave."

(4) Professor T. J. J. SEE: "The dimensions and density of Neptune."

(5) Professor E. FRISBY: "Some remarkable properties of recurring decimals."

(6) Professor G. T. SELLEW: "History of the complex number."

(7) Professor JAMES MCMAHON: "The expression of a rational polynomial in a series of Bessel Functions of the *n*th order."

(8) Dr. G. A. MILLER: "Report on the groups of an infinite order."

(9) Professor L. E. DICKSON: "Definition and examples of Galois fields."

(10) Mr. W. B. FITE: "On the metabelian groups whose invariant operators form a cyclical subgroup."

(11) Mr. H. E. HAWKES: "Note on Benjamin Pierce's linear associative algebra."

(12) Mr. F. H. LOUD: "Sundry metrical theorems connected with a special curve of the fourth class."

(13) Mr. G. A. HILL: "Variations of latitude."

(14) Mr. G. A. HILL: "The comparative accuracy of the transit circle and the vertical circle."

(15) Professor J. N. STOCKWELL: "New light on ancient eclipses."

(16) Professor J. N. STOCKWELL: "Secular variations of the motions of the planets."

(17) Professor J. N. STOCKWELL: "A new method of computing the Laplace coefficients of planetary pertubation."

(18) Miss MARY PROCTOR: "Miss Catharine Wolf Bruce."

(19) Miss M. E. TRUEBLOOD: "The directive force of philosophy upon mathematics."

(20) Mr. W. G. LEVISON: "On a method of photographing the entire corona, employed at Newberry, S. C., for the total solar eclipse, May 28, 1900."

Two of the titles that appeared on the programme are not reproduced here since these papers did not reach the secretary in time to be presented at the meeting. In the absence of their authors the papers by Professor McMahon and Mr. Fite were presented by Professor Crawley and Dr. Miller respectively. Mr. Hawkes' paper was read by title. The two papers by Mr. Hill will be published in the *American Journal of Science* and in *Science* respectively. Dr. Miller's report will appear in the BULLETIN. Abstracts of the other papers, with the exception of the last three, are given below.

President H. S. Pritchett, lately Superintendent of the U. S. Coast and geodetic survey, considered the three questions: "What is the purpose of this service? Is it properly organized to carry out this purpose? What lines of work should be followed to accomplish the purpose in view?" The first and principal work of the Coast and geodetic survey is the hydrographic survey of the coasts of the United States and the islands under her jurisdiction. In hydrography and topography a part of the energy of the service will go to the resurvey of parts of the mainland of the United States; but by far the greater effort will go to the hydrographic surveys of Porto Rico, Hawaii, the Philippine Islands, and Alaska. In geodesy, the most interesting work of the next eight years will be the completion of an arc extending along the ninety-eighth meridian from the Rio Grande to the Canadian border. By the coöperation of Mexico and of Canada it is expected that this arc will ultimately have for its southern limit a latitude of about 19 degrees and for its northern terminus the latitude of nearly 70 degrees. The completion of the precise level net for the United States, and the differential gravity determinations now being carried out in connection with similar work by other nations, are the most important other objects of this service. The work of the Division of terrestrial magnetism contemplates a general magnetic survey of the country and of the waters adjacent within the next ten years.

Mr. Hayford's paper is in abstract as follows: The net of precise level lines recently adjusted in the Computing Division of the Coast and geodetic survey comprises over 12,000 miles of spirit leveling, and 1,400 miles of water leveling, run by the Coast and Geodetic Survey and other organizations. The manner of making several preliminary "studies" before making the final least square adjustment was explained. The points requiring especial attention were, the assignment of relative weights to different classes of leveling, the attempt to detect systematic errors, and the assignment of the relation of weight to length of line. The least square method of computation was used throughout and the assumptions upon which it is based were checked, as far as possible, by direct appeals to the facts. The only well-marked form of a systematic error found, is one which is due to changes of temperature in the instrument during the period of observation at a station. The instrument shown was one which is about to be put into use in the Coast and geodetic survey; it is designed with special reference to eliminating the systematic error just referred too, and to furnish rapid observations of the highest degree of accuracy.

The object of the first paper by Professor See, was to direct attention to certain points in the theory of the propagation of the tide wave. After defining the tide wave by the general relations for horizontal and vertical motion,

$$x' = x + \varphi(x_1 t), \quad y' = y + \psi(\ _1 t),$$

which are transformed into

$$\Xi = A\left(e^{\frac{2\pi\gamma}{\lambda}} + e^{-\frac{2\pi\gamma}{\lambda}} \right) \cos\,(nt - \nu x),$$

$$H = -A\left(e^{\frac{2\pi\gamma}{\lambda}} - e^{-\frac{2\pi\gamma}{\lambda}} \right) \sin\,(nt - \nu x),$$

γ being the depth of the sea, λ the wave length, and ν any integral number, the author shows that in case of the tides, where λ is large compared to γ, the path of any particle is an elongated ellipse, giving a horizontal motion about 1,000 times greater than the vertical motion. He then gives the tide-generating potential in the usual form

$$V = \frac{3}{2}\frac{m\nu^2}{\rho^3}\left(\cos^2\omega - \frac{1}{3}\right),$$

and shows by derivation the lengths of the four arcs into which the circumference is divided, two being each 109°.5 in length, the other two 70°.5. A figure showing the motion of the water in every part of an equatorial canal at any instant, indicates clearly why the wave advances when once started, as the flow is forward in the two long arcs, and backward in the short arcs. The author states that the tide wave is at first a forced wave, afterwards acting as a free wave ; but this view of the propagation holds for both classes of oscillations, when treated separately, and is approximately true when the two classes of waves are compounded.

In his second paper Professor See treats historically the question of the diameter of Neptune, showing that the earliest measures, made in 1846, place it at about 3″, and that subsequent measures range from 4″.3 to 2″.00. With every improvement in the definition of telescopes, the diameter has shrunk. He then showed, from observations made in 1899–1900 with the great equatorial of the Naval Observatory under better conditions than have been enjoyed by previous observers, that the true diameter of the planet is about 2″.00, or 27,190 miles. The diameter hitherto accepted in standard works on astronomy is 34,000 miles. The former density was 1.11 that of water, but this is now doubled and put at 2.26.

Professor Frisby gave some brief special methods of combining numbers, and especially of finding the recurring decimals which represent rational fractions.

Professor Sellew traced the history of the complex number from the first appearance of the imaginary in the Stereometrica of Hero of Alexandria in the second century B. C. The idea that the imaginary must be in the plane without the line of real numbers was first given by Wallis (1685). Kühn's construction (1753) introduced the idea of direction. Wessel (1797) was the real inventor of the modern representation of the complex number. His analytic representation of direction was the best before the time of Hamilton.

It is known that in certain cylindrical boundary problems it is necessary to expand $f(r, \theta)$ in a Fourier series of the form

$$\sum_{n=1}^{\infty} \varphi_n(r) \cos n\theta,$$

and then to expand $\varphi_n(r)$ in an nth order Bessel series of the form

$$A_1 J_n(\lambda_1 r) + A_2 J_n(\lambda_2 r) + \cdots,$$

in which $\lambda_1, \lambda_2, \cdots$ are determined from the roots of a certain equation to be satisfied on the boundary $r = a$, and the coefficients are given by

$$A_s = \frac{\int_0^a \varphi_n(r) . r J_n(\lambda_s r) dr}{\int_0^a r [J_n(\lambda_s r)]^2 dr}.$$

The value of the definite integral in the denominator is given in the text-books, but no method of evaluating the integral in the numerator has been given. The object of the paper by Professor McMahon is to reduce this integral in the commonest case, viz., that in which $\varphi_n(r)$ is expressible in a series of powers of r with integer exponents (positive or negative). For this purpose formulæ are given by means of which $\int x^m J_n(x) dx$ is made to depend on $\int x J_n(x) dx$ and $\int J_n(x) dx$, and ultimately on $\int J_0(x) dx$ alone.

Thus the complex numerical treatment of a large class of cylindrical harmonic problems requires the introduction and tabulation of a new fundamental transcendent, which may be defined by the equation

$$H_0(x) = \int_0^x J_0(x) dx.$$

This function may be tabulated either from the power-series, or from the relation

$$H_0(x) = 2[J_1(x) + J_3(x) + J_5(x) + \cdots].$$

Professor Dickson developed the definition of Galois field as follows: Let p be a prime number and $P(x)$ a polynomial of degree n in x having integral coefficients, and suppose $P(x)$ to be *irreducible* modulo p, *i. e.*, suppose it to be impossible to give $P(x)$ the form

$$P(x) = P_1(x) \cdot P_2(x) + p \cdot P_3(x),$$

where the $P_i(x)$ are polynomials having integral coefficients, the degrees of $P_1(x)$ and $P_2(x)$ being $<n$. If we divide any polynomial $F(x)$ having integral coefficients, by the function $P(x)$, we obtain a quotient $Q(x)$, and a remainder which can be written

$$f(x) + p \cdot q(x), \quad f(x) \equiv a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-2},$$

where every a_j is a positive integer $<p$. Hence

$$F(x) \equiv f(x) + P(x)\,Q(x) + p \cdot q(x).$$

We say $f(x)$ is the residue of $F(x)$ modulo p and $P(x)$ and write

$$F(x) \equiv f(x) \quad [\text{mod } p, P(x)].$$

The totality of functions $F(x)$ obtained by giving to the polynomials $Q(x)$ and $q(x)$ all possible forms is said to constitute a class of residues. Two polynomials

$$F_i(x) = f_i(x) + P(x)\,Q_i(x) + p . q_i(x) \qquad (i = 1, 2)$$

belong to the same or different classes according as f_1 and f_2 are identical or not. Each a_j in $f(x)$ having p values, there are evidently p^n distinct classes of residues. The class to which $F_1 \pm F_2$ or $F_1 \cdot F_2$ belongs depend merely upon the functions $f_1 \pm f_2$ or $f_1 \cdot f_2$, being independent of the Q_i and q_i. Hence classes of residues combine unambiguously under addition, substraction and multiplication. Furthermore, an arbitrary class C_{F_2} may be divided by any class C_{F_1}, not the class zero, yielding as quotient a unique third class. In fact, p being prime and $P(x)$ irreducible modulo p, and $F_1(x)$ being not divisible by $P(x)$ modulo p, we can determine, by a process analogous to that for finding the greatest common divisor, integral functions $F_1'(x)$ and $P'(x)$ such that

$$F_1'(x) \cdot F_1(x) - P(x) \cdot P'(x) \equiv 1 \quad (\text{mod } p).$$

Hence the function $F_1'F_1$ belongs to the class unity. It follows that

$$\frac{C_{F_2}}{C} = \frac{C_{F_2} C_{F_1'}}{C_{F_1} C_{F1}'} = C_{F_2 F_1'}.$$

The p^n classes of residues modulis p and $P(x)$ therefore form a field, called a Galois field of order p^n.

The main object of Mr. Fite's paper was to demonstrate the following theorems :

THEOREM I. If the commutators of a metabelian* group G of order p^m, p being a prime, form a cylical group of order p^a , then the number of independent generators of any given order of the group of cogredient isomorphisms G' must be even.

THEOREM II. There are exactly $\beta + 1$ groups of order $p^{a+2\beta s}$, p being an odd prime, s being any positive integer, and $\beta \gtreqless a$, such that their invariant operators form a cylical group of order p^a and that their groups of cogredient isomorphisms are abelian of the type (β, β, $\cdots$ to $2s$ terms).

THEOREM III. The number of metabelian groups whose invariant operators form a cyclical group of order p^a and whose groups of cogredient isomorphisms are of the type (a_1, a_1 ,$\cdots$, to $2s_1$ terms ; a_2, a_2 $\cdots$ to $2s_2$ terms ; $\cdots$; a_n , a_n ,$\cdots$ to $2s_n$ terms), where s_1, s_2, $\cdots$, s_n are positive integers and $a_1 < a_2 < a_3 < \cdots < a_{n-1} < a_n$, is $(a_1+1)(a_2-a_1+1)(a_3-a_2+1)$

$$\cdots(a_n-a_{n-1}+1).$$

The curve studied by Mr. Loud is the parallel to the hypocycloid of four cusps, and is briefly noticed by Salmon in his Higher plane curves, §§ 118 and 119. The present paper gives a simple method of construction by points and establishes several theorems connecting the curve with the metric geometry of n lines in a plane. The treatment adopted employs the circular coördinates x and y, equal respectively to $x + iy$ and $x - iy$ in the Cartesian system. Some of the more obvious properties of the curve, as well as its varieties of form, are deduced. The methods and results given by Professor Morley in the second number of the *Transactions* have been extensively employed.

In his first paper Professor Stockwell argues that ancient eclipses throw much more light on ancient chronology than might be inferred from the paper by Professor McFarland which was read before this Section at the Columbus meeting. The second paper is intended to show that there are secular equations of the mean distances of the planets from the sun. The third paper is in abstract as follows :

The Laplace coefficients of planetary perturbation arise from the development of the function

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* If the commutators of a group are self-conjugate the group is called *metabelian*. The group of cogredient isomorphisms of such a group is abelian, and conversely.

$$\{1 + a^2 - 2a\cos\beta\}^{-s}$$

in an infinite series depending on the cosines of β and its multiples. If we assume that

$$\{1 + a^2 - 2a\cos\beta\}^{-s} = \tfrac{1}{2}b_s^{(0)} + b_s^{(1)}\cos\beta + b_s^{(2)}\cos 2\beta + b_s^{(3)}\cos 3\beta + \cdots + b_s^{(i)}\cos i\beta, \qquad (1)$$

the coefficients $b_s^{(i)}$ are the Laplace coefficients required.

If we put

$$\frac{2a}{1 + a^2} = a', \qquad (2)$$

we shall have

$$\{1 + a^2 - 2a\cos\beta\}^{-s} = (1 + a^2)^{-s}\{1 - a'\cos\beta\}^{-s}; \qquad (3)$$

and if we now assume that

$$\{1 - a'\cos\beta\}^{-s} = \tfrac{1}{2}\dot{b}_s^{(0)} + \dot{b}_s^{(1)}\cos\beta + \dot{b}_s^{(2)}\cos 2\beta + \dot{b}_s^{(3)}\cos 3\beta + \cdots + \dot{b}_s^{(i)}\cos i\beta, \qquad (4)$$

we shall have

$$b_s^{(i)} = \dot{b}_s^{(i)} \cdot \{1 + a^2\}^{-s}. \qquad (5)$$

If we now develop the first member of equation (4) by the binomial theorem, we shall have an infinite series of terms which may readily be computed by means of a table of logarithms of the coefficients of the different powers of cos β, together with a similar table of logarithms of the coefficients $\cos^n\beta$ when it is developed in terms of the cosines of β and its multiples.

Tables containing the logarithms of the coefficients of these functions have been computed, so that we may easily compute all values of $\dot{b}_s^{(i)}$ for all values of $i = 0$, to $i = 35$. And when $\dot{b}_s^{(i)}$ has been found $b_s^{(i)}$ may be found by equation (5).

G. A. MILLER.

CORNELL UNIVERSITY.