The principal operations in this calculus consist in developing both members of the equation

$$Ge^{(x+a)u+(y+b)v+\cdots} = \varphi(x+a,\, y+b,\, \cdots)$$

according to powers of the increment a, and equating the coefficients of equal powers. By this process the following fundamental formula is found:

$$Ge^{xu+yv+\cdots}\psi(u,\, v,\, \cdots) = \psi(D_x,\, D_y,\, \cdots)\varphi(x,\, y,\, \cdots).$$

In the applications the function ψ has to be developed according to powers of $D_x = \frac{d}{dx}$, $D_y = \frac{d}{dy}$, $\cdots$, after which the differentiations are to be performed.

The idea of this calculus seems to be to replace a given function $\varphi(x, y, \cdots)$ by the exponential function $Ge^{xu+\cdots}$, and then, after some suitable transformations, to return from the exponential to the given function. This suggests the idea that G is an operator like Σ, although the author avoids saying so, perhaps owing to the objections which had been raised against Liouville's assumption, that every function can be developed into an exponential series. The work of Oltramare was recently reviewed by Professor Lovett in the BULLETIN.

GEORGETOWN UNIVERSITY,
May, 1900.

BURNSIDE'S THEORY OF GROUPS.

Theory of Groups of a Finite Order. By W. BURNSIDE,* M.A., F.R.S.; Professor of Mathematics at the Royal Naval College, Greenwich. Cambridge, The University Press, 1897. 8vo., xvi + 388 pp.

THIS work enjoys the distinction of being the first treatise on the theory of groups which does not consider the applica-

* The joint author of Burnside and Panton's "Theory of Equations" is W. S. Burnside, an Irishman, professor of mathematics and fellow of Trinity College, Dublin. He is not related to the author of the work here considered, who is of Scotch extraction and has not published any other book. He has, however, published a number of memoirs—most of which relate to the theory of groups—in the *Proceedings of the London Mathematical Society* and in the *Messenger of Mathematics*. It seemed desirable to mention these facts since the two authors—William Burnside and William Snow Burnside—are frequently confused. Cf. Netto's review, Schlömilch's *Zeitschrift*, vol. 44 (1899), p. 20.

tions. As may be inferred from the title, the author has practically confined his attention to the groups of a finite order. The known regions which are bounded by these restrictions are, at the present time, not too extensive to be described in one volume. Happily, our author is so familiar with these regions that he does not confine himself entirely to leading the reader by known roads to the many interesting objective points. He has pointed out many short routes as well as a number of new objects of interest.

The theory of groups of a finite order (under the form of substitution groups) was first developed as a branch of the theory of equations; and the works of Lagrange,* Abel, and Galois on the solution of equations have contributed most powerfully towards its early development. In recent years it has been pointed out by Jordan, Klein, Cayley, and others that this subject has extensive geometrical applications. This has led to the study of groups in the abstract; *i. e.*, independent of any particular mode of representation. The work before us aims to treat the subject in this modern spirit.

Nevertheless considerable space is devoted to substitution groups. This may possibly be partly due to the fact that the subject was first developed from the standpoint of substitution groups. The author states in the preface that it is done because, "in the present state of our knowledge, many results in pure theory are arrived at most readily by dealing with properties of substitution groups." The concepts substitutions and group are so closely related that the former seems a natural precursor of the latter; and the study of many of the group properties, such as the group of isomorphisms and the transformation of conjugate subgroups and operators, seems to lead naturally to substitution groups. A knowledge of substitution groups can thus be utilized very frequently in the study of the abstract group properties, even if it is not essential for the study of these properties.

A number of interesting theorems such as those which relate to primitivity, imprimitivity, transitivity, and intransitivity apply only to substitution groups. Many readers would regret to see these landmarks of early triumphs banished from the theory to which they led. To such readers the method followed by our author will be entirely satisfactory, while those who have confined their attention to the more abstract theories might have preferred to find fewer

* Pierpont, "Lagrange's place in the theory of substitutions," BULLETIN, 2d series, vol. 1 (1895), p. 196.

substitution group illustrations—especially in the early chapters of the work. The notation of substitution groups has, however, been excluded from the proofs and investigations of chapters II. to VII., in which most of the fundamental properties have been developed.

The value of the book is increased by a number of carefully selected examples. These examples are generally chosen with a view towards continuing or completing the discussions in the text rather than with a view towards illustrating the theoretical developments. Consequently many of them seem too difficult to be taken up at the first reading. Some of them have not been stated with sufficient care: *e. g.*, number 2 on page 10 is evidently not true when $m = 1$ and $n > 2$; and the well known quaternion group is an exception to number 4, page 195, which is referred to Dyck. It may be remarked that the group of order 12 which contains six operators of order four is an exception to Dyck's theorem (*Mathematische Annalen*, vol. 22, p. 101) from which the given example seems to have been obtained. The second part of example 2, page 89, cannot be proved, since it is easy to construct groups of the specified type which are not generated by two operators that satisfy the given conditions.

As the book is written for the beginner, "to introduce to the reader the main outlines of the theory of groups of finite order without any applications," the first three chapters are devoted to the most elementary matters—the explanation of the notation of substitutions, the definitions and elementary illustrations of the terms group, subgroup, direct product,* etc., and a few fundamental theorems. The theorem stated on page 22 to the effect that every group of a finite order can be represented as a regular substitution group is not due to Dyck, as stated in the table of contents. It was given explicitly by Capelli at an earlier date (*Giornale di Matematiche*, 1878, p. 45) and somewhat less explicitly by Jordan at a still earlier date (Traité des substitutions, 1870, p. 60). The symbol G/Γ used to represent the quotient group of G with respect to its self-conjugate subgroup Γ seems to be due to Jordan (*Bulletin de la Société Mathématique*, vol. 1 (1873), p. 46). It is certainly not due to Hölder as is stated on page 38.†

* This definition of direct product does not appear to be as simple as possible; cf. *Transactions*, vol. 1 (1900), no. 1, p. 66. It may be remarked that Burkhardt gives an incomplete definition of direct product in his article on "Endliche discrete Gruppen," *Encyk. der math. Wissenschaften*, vol. 1, p. 219.

† Netto makes the same error in his new Algebra, vol. 2, 1900, p. 343.

A number of minor inaccuracies, which the reader will readily discover, occur in these as well as in the later chapters. Such inaccuracies tend to confuse the beginner, who is so prone to study the book, instead of using it as a tool to study the subject. For instance, on page 16, line 4, it is stated that the order of the product of two commutative operators is the least common multiple of the orders of these operators. That this is not always true may be seen from each of the following facts: the product of the two commutative substitutions $(abcd)(efgh)$ and $(afch)(bgde)$ is of order two, and the product of an operator and its inverse is identity. The statement is evidently always true when the groups generated by the two commutative operators have only identity in common. This condition is sufficient but it is not necessary. The interchange of the substitutions represented by D and E, on page 19, example VII., is an error of less importance.*

Chapter IV. is devoted to abelian groups. This subject is so important that a more extensive exposition would doubtless have proved agreeable to a large number of the readers. Several of the most interesting theorems are given as examples at the end of the chapter; but most beginners will probably be unable to prove them within a reasonable length of time. After a few preliminaries the author proves that any abelian group of order p^α (p being any prime number) contains a set of independent generators and that the orders of these generators are invariants of the group. This is followed by some very interesting investigations in regard to the subgroups of abelian groups of order p^α. Unfortunately the treatment of this subject is not always as complete as might be desired and there seem to be a few inaccuracies. For instance the notation of the latter part of theorem III., p. 58, seems to me to imply that a group of type (5, 2) contains a subgroup of type (3, 3), which is evidently impossible.

In chapter V. we meet the El Dorado of the theory of groups,—the groups of order p^α, p being any prime number. Cauchy proved that every group whose order is divisible by p must contain one or more subgroups of order p. More than twenty-five years later Sylow proved that a group contains one and only one system of conjugate subgroups of order p^α whenever its order is divisible by p^α but not by $p^{\alpha+1}$. These results indicated the great importance of a careful study of the groups of order p^α. Fortunately they give rise

* Mr. Kuhn called my attention to this interchange of substitutions.

to a large number of very interesting and unusually simple theorems. Many of these are old, but a considerable part of them are of recent origin being largely due to the investigations of Frobenius and Burnside. The general theorem published in the *Messenger of Mathematics*, vol. 27, 1897, appeared too late to be included in the present work.

The explicit theorems of this chapter together with those which are brought out by the illustrative examples give a comprehensive view of the groups of order p^a. The examples are drawn, to a large extent, from the publications of Hölder, Young, and Cole and Glover. On pages 68 and 69, Burnside undertakes the determination of every group G of order p^{r+s} in which the subgroup H_1 which is composed of the self-conjugate operators is the cyclical group of order r, while G/H_1 is an abelian group of type (1, 1, ···, with s units). He concludes that the number of these groups is $2^{\frac{1}{2}s}$, which is evidently incorrect.*

Chapter VI. is devoted to Sylow's theorem and the important extension due to Frobenius. For abelian groups this theorem was proved in Chapter IV. and the developments of the present chapter are somewhat simplified thereby. The fundamental importance of the theorem is clearly brought out by a number of carefully selected examples. In one of these examples all the possible groups of order 24 are determined. All these groups had been determined somewhat earlier,† but no reference is given to the earlier publications on this subject. The reference on p. 105 in regard to the published abstract groups is incomplete since all the groups whose order is less than 48 were known.‡ In giving the extension of Sylow's theorem the author follows Frobenius quite closely.

The composition series of a group, which is considered in Chapter VII., plays a very important rôle not only in the theory of groups but also in the Galois theory of equations. The foundation for the developments along this line was laid by Jordan in his article published in *Liouville*, vol. 14 (1869), p. 139. The reference on p. 119 should be to this article, instead of to Jordan's Traité des substitutions Here Jordan proved the very important theorem that the factors of composition of a given group are invariant. Hölder has extended these results somewhat by proving that the factor groups are also invariant. Burnside follows Hölder's developments quite closely in the first part of the

* Mr. Fite called my attention to this error.

† *Quar. Jour. of Math.*, vol. 28 (1896), p. 274.

‡ *Ibid.*, p. 32.

chapter, and he terminates it with a few theorems on soluble groups followed by several examples. In his recent Algebra, vol. 2, 1900, Netto uses chief series (Hauptreihe) in a somewhat different sense from Burnside, hence his example, p. 338, does not substantiate the criticism which Netto makes in the foot-note on p. 337 in regard to Burnside's theorem given on p. 123 of the present work.

The next three chapters (VII.–X.) are devoted to the theory of substitution groups. Only a brief sketch of this theory could be given in the limited space, especially since a great part of it is devoted to the determination of all the possible substitution groups of given degrees. The errors which occur in these enumerations have been noted elsewhere.* On p. 189 an error which occurs in Jordan's Traité (p. 65) is pointed out without noting that Jordan had corrected the error himself soon after the appearance of his work. The beautiful theorems of Jordan in regard to the limit of transitivity, which were published in the same article † as the corrections mentioned, would probably have been more useful than the one given on p. 152.

The material of these chapters seems, in general, to have been wisely selected. The many illustrative examples enable the student to get a clear insight into the real meaning of the terms transitivity, intransitivity, primitivity, and imprimitivity. The construction of intransitive groups furnishes a good means to acquire a thorough working knowledge of quotient groups and isomorphisms. The references in these chapters are not as complete as those of most of the other chapters. In some of the illustrative examples only a brief outline is given, so that the reader has to supply a number of intermediate statements to complete the proofs. Corollaries II., on pp. 177 and 184, evidently do not apply to the primitive groups of a prime order.

The principle of isomorphisms is one of those thought saving developments upon which special stress has been laid in recent years. In chapter XI. the isomorphisms of a group with itself are considered. The beginning of this chapter is not auspicious, for in the second paragraph it is stated that a group of order 2 is the only one in which each operator corresponds to itself when we let the inverse of each operator of a group correspond to itself. This evident inaccuracy is, however, followed by one of the best and most interesting chapters of the entire volume. The devel-

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* BULLETIN, vol. 5 (1899), p. 249.

† Jordan, *Bull. de la Soc. Math. de France*, vol. 1 (1873), p. 41. Cf., BULLETIN, vol. 4 (1897), p. 144.

opments would doubtless have been simplified by giving some other methods of making a group isomorphic with itself, but the entire subject matter is so new that a great degree of simplicity could scarcely be expected.

In this chapter Burnside introduces a very useful new term (holomorph) which enables him to employ an interesting theorem in regard to regular groups, due to Jordan,* in the study of the group of isomorphisms. It would perhaps have been better to define the holomorph of a regular group G as the substitution group which includes all the possible substitutions in the elements of G that are commutative with G instead of the abstract group which is simply isomorphic to this substitution group. The largest subgroup of the given substitution group that does not involve a given element is evidently simply isomorphic to the group of isomorphisms of G.† In article 171, on the group of isomorphisms of an abelian group of order p^n and type (1, 1, ⋯, to n units), reference should have been given to Moore's earlier article bearing on the same subject ‡ and the theorem that the symmetric group of n elements is a complete group, except when $n = 6$, should have been referred to the BULLETIN, vol. 1, p. 258, as well as to *Mathematische Annalen*, vol. 46, p. 345.

Chapters XII. and XIII. are devoted to the graphical representation of a group and to Cayley's color groups. Here we meet for the first time some groups of an infinite order. The first chapter is, to a great extent, a reproduction of Dyck's well known article which was published in the *Mathematische Annalen* in 1882, vol. 20. The first volume of the *American Journal of Mathematics* contains a brief note on color groups by Cayley. Recently Maschke published a long article on this subject in the same journal, vol. 18, 1896, pp. 156–188. Burnside gives only a brief sketch of these groups, devoting merely five pages to them. Chapter XIV. contains a very brief sketch of the linear group, which forms the subject matter of the greater part of Jordan's Traité des substitutions. In the analysis of the fractional linear groups Gierster's noted memoir, published in volume 15 of the *Mathematische Annalen*, is closely followed. Moore has called attention § to the fact that the proof given on page 339, to the

* Traité des substitutions, 1870, p. 60.

† On page 221, vol. 1, of the *Encyclopädie der mathematischen Wissenschaften*, it is stated that the group of isomorphisms of G consists of those substitutions in the same elements that are commutative with G. This is evidently incorrect; cf. BULLETIN, vol. 5 (1899), p. 245.

‡ Moore, BULLETIN, vol. 2 (1895), p. 33

§ Moore, *Math. Annalen*, vol. 51, p. 419.

effect that the group of isomorphisms of the group of order 16 which has four independent generators is simply isomorphic to the alternating group of degree 8, is incorrect; and that the proper references are lacking in regard to the developments on pp. 336–339 and p. 342.

In the final chapter (XV.) of the book the soluble and composite groups are studied. In speaking of this chapter the author says (p. 334): "if the results appear fragmentary, it must be remembered that this branch of the subject has only recently received attention; it should be regarded rather as a promising field of investigation than as one which is thoroughly explored." Sylow's proof that every group whose order is p^a is soluble was one of the first general results in this direction. Recently Frobenius succeeded in establishing an equally general and interesting theorem by proving that all the groups whose order is not divisible by the square of a prime number are soluble. A large number of theorems of less extensive application have recently been published. These are clearly set forth in the present chapter, which seems more replete with recent results and problems awaiting solution than any other.

In conclusion we would say that the book seems to us to be a strong one notwithstanding the fact that the inaccuracies are somewhat numerous.* We have endeavored to call attention to a sufficient number of these to put the reader on his guard not to accept the results without a careful examination of the proofs. It is to be hoped that the demand for the book will warrant a new and carefully revised edition. A work containing such a rich store of facts, and nothing but facts, would be exceedingly valuable. It should be remembered that some of the mathematical works which have had a great influence on the developments in certain directions have been greatly marred by errors, so that we should not allow the minor defects of the present work to blind us to its many merits.

The student of the theory of groups will find here a rich storehouse of material, and the investigator will find numerous suggestions in regard to problems which await solution and methods of attacking them. The very errors will doubtless serve as a starting point for many investigations, since there are few things that can furnish a stronger impetus to many beginners than the thought of correcting a noted mathematician. It is therefore to be hoped that the book will be a means to arouse "interest among English (speaking) math-

* The book contains no references to Italian sources, although Italy has contributed very materially towards the development of this subject.