THE APRIL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

A REGULAR meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York City on Saturday, April 28, 1900. The American Physical Society was in session at the same time, and a part of the afternoon was devoted to a joint meeting, at which the papers noted in the list below were presented by Professors E. W. Brown and R. S. Woodward. The attendance during the day was about forty, including the following twenty-four members of the Society:

Professor E. W. Brown, Professor F. N. Cole, Dr. W. S. Dennett, Professor T. S. Fiske, Mr. A. S. Gale, Dr. G. B. Germann, Dr. A. A. Himowich, Mr. S. A. Joffe, Mr. C. J. Keyser, Dr. James Maclay, Dr. Emilie N. Martin, Dr. G. A. Miller, Mr. C. A. Noble, Professor M. I. Pupin, Professor J. K. Rees, Mr. C. H. Rockwell, Professor Charlotte A. Scott, Dr. Virgil Snyder, Professor E. B. Van Vleck, Professor L. A. Wait, Professor A. G. Webster, Dr. J. Westlund, Miss E. C. Williams, Professor R. S. Woodward.

The President of the Society, Professor R. S. Woodward, occupied the chair, yielding it during the joint meeting to Professor William Hallock of the Physical Society. The amendments of the Constitution outlined in the report of the February meeting were adopted. The Council announced the election of the following persons to membership in the Society: Professor R. D. Ford, St. Lawrence University, Canton, N. Y.; Dr. L. W. Reid, Princeton University, Princeton, N. J. Eight applications for membership were reported.

The following papers were read at this meeting:

(1) Dr. VIRGIL SNYDER: "On some invariant scrolls in collineations which leave a group of five points invariant."

(2) Mr. A. S. GALE: "Note on four theorems of Chasles."

(3) Professor CHARLOTTE A. SCOTT: "A theorem on quadrilaterals in space" (preliminary communication).

(4) Mr. F. H. LOUD: "Sundry theorems concerning n-lines in a plane."

(5) Dr. E. J. WILCZYNSKI: "Transformation of systems of linear differential equations."

(6) Professor FLORIAN CAJORI: "Semi-convergent and divergent series whose product is absolutely convergent."

(7) Professor E. W. BROWN: "A possible explanation of the eleven year period of sunspot activity."

(8) Professor R. S. WOODWARD: "An elementary method of integrating certain linear differential equations."

(9) Dr. G. A. MILLER: "On a certain class of abelian groups."

(10) Professor H. B. NEWSON: "On singular transformations and continuous groups."

(11) Professor E. O. LOVETT: "Group theory and geometry of four dimensions."

(12) Professor E. O. LOVETT: "The condition that a linear total differential equation be integrable."

(13) Professor C. H. HINTON: "Observations on the principle of duality."

Professor Hinton's paper was presented to the Society through Professor Lovett. In the absence of the authors, Mr. Loud's paper was read by Professor C. A. Scott, and the papers of Dr. Wilczynski, Professor Cajori, Professor Newson, Professor Lovett, and Professor Hinton were read by title. Abstracts of the papers are given below.

Dr. Snyder's paper is in abstract as follows: There are six cyclical collineations of space whose orders are less than seven: T_2, the harmonic homology; T_3, the axial involution; $T_4 \equiv (123)(4)(5)$, in which the elements of space are in triads; $T_5 \equiv (123)(45)$, in sextuples; $T_6 \equiv (1234)(5)$, in quadruples; and $T_7 \equiv (12345)$, in quintuples. A cyclical collineation of order six and having five elements chosen arbitrarily does not exist. Systems of scrolls contained in a linear congruence which are left invariant by these transformations are discussed, with particular reference to their asymptotic lines. The transformations T_3, T_4, and T_7 leave a [3, 1] scroll invariant in such a way that each asymptotic line is transformed into itself. The quartic surface, locus of the vertex of a quadric cone which passes through two triads of T_4, goes into itself; it touches a [3, 1] scroll in every point of the twisted cubic through the six points. This cubic is an asymptotic line on both surfaces. Every [3, 1] scroll having a pair of double imaginary pinch points has asymptotic lines of order 3.

In 1860 Chasles published the theorems that, if a right line be displaced, the middle points of the chords joining congruent points lie on a line or are identical, and that the planes perpendicular to these chords at their middle points pass through a line or are identical; and also corresponding theorems relating to the displacement of a plane figure.

These theorems form the basis of the geometrical consideration of the theory of finite displacements in space as developed by Study. Mr. Gale's note, which will appear in the *Annals of Mathematics*, shows how the theorems of Chasles follow from Wiener's theorem that every displacement in space may be resolved in ∞^2 ways into the product of two axis reflections. By means of this theorem, displacements are resolved into the product of two transformations such that the points of an arbitrary line or plane are invariant with respect to the first transformation. The theorems of Chasles follow immediately from properties of the second transformation.

The following is a summary of Professor Scott's preliminary communication: If corresponding vertices a, a'; b, b'; c, c'; d, d'; e, e'; f, f' of two quadrilaterals in different planes are joined, an interesting configuration of 6 lines $\alpha, \beta, \gamma, \delta, \varepsilon, \varphi$ is obtained. It is cut by a singly infinite system of planes in complete quadrilaterals; these planes, being common tangent planes to two ruled quadrics with a common generator, are tangent planes to a cubic torse, and consequently also to a twisted cubic. The configuration is intimately associated with the theory of cubic curves and torses, yet there seems to be no recognition of the theorem in the literature of this subject. There is no difficulty in obtaining a proof in various forms; but to exhibit the theorem as fundamental in the treatment of projective forms, a proof that shall rely solely on the formal properties of points, lines, and planes is desirable. It appears that such a proof must be obtainable by means of triangles in perspective; but it still remains to be found.

The point of departure for Mr. Loud's paper is furnished by Professor Morley's article in the April *Transactions* "On the metric geometry of the plane n-line." Professor Morley, having discussed a certain class of theorems resulting from combining n-lines by twos, had pointed out that they might also be combined three or more at a time with analogous results. The present paper considers the case of combination by threes. After briefly noting those of his results which are strictly analogous to some of Professor Morley's, the author has indicated a means of deducing one series of theorems from the other. A distinction between the two series is pointed out in the fact that in the present case each line is restricted to a single direction. An inquiry into the relation of two figures, derived one from the other by the reversal of one given line, follows, and occupies the greater part of the paper. The general nature of the theorems discussed is exemplified in the following: The theorem which in the parallelism of the two series becomes the analogue of Miquel's asserts that the twenty points, which are centers of circles tangent to six given directed lines taken by threes, lie by fours on fifteen circles, whose circumferences meet by fives in six points, and these points lie on a single circle. Again, as an illustration of the effect of reversal, we have the theorem: If all but one of an even number of given lines be turned, each through a right angle, about the point at which it meets the one remaining line, and if the Clifford circles be constructed foı these rotated lines in both their original position and that to which they are turned, these circles intersect orthogonally. By making each line in succession the one upon which the others are turned, as many orthogonal pairs of circles are obtained as there are given lines, and all these circles pass through a common point, which is the focus of the p-fold parabola tangent to all the given lines.

Dr. Wilczynski's paper furnishes a proof of the following theorem: The most general transformation which converts a general system of linear homogenous differential equations into another system of the same form and order is

$$x = f(\xi), \quad y_k = a_{k1}(\xi)\eta_1 + \cdots + a_{kn}(\xi)\eta_n \quad (k = 1, 2, \cdots, n),$$

where f and a_{ki} are arbitrary functions of ξ. This theorem is the basis of a theory of invariants of such systems upon which the author is now engaged.

Pringsheim states in one of his articles that he sees no *a priori* reason why one might not be able to construct semi-convergent series possessing properties which would render the product of two series absolutely convergent. But he was unable to produce examples of such series. In the present paper Professor Cajori gives not only pairs of semi-convergent series, but also pairs of series, one semi-convergent, the other divergent, whose product converges absolutely. The process of constructing the series is explained. In all cases brought under consideration, one of the factor series has zero for its sum. Example: Of the following two series, T_1 is semi convergent, T_2 is divergent; their product is absolutely convergent:

$$T_1 \equiv \sum_{p=0}^{p=\infty} \left(\frac{1}{6p+1} - \frac{1}{6p+2} + \frac{1}{6p+3} - \frac{1}{6p+1} + \frac{1}{6p+2} - \frac{1}{6p+3} \right),$$

$$T_2 \equiv \sum_{p=0}^{p=\infty} \left(\frac{1}{6p+1} + \frac{1}{6p+2} - \frac{1}{6p+3} + \frac{1}{6p+1} + \frac{1}{6p+2} - \frac{1}{6p+3} \right).$$

The following differential equations are of frequent occurrence in mechanics and mathematical physics :

$$\frac{dx}{dt} + \alpha y = 0, \quad \frac{dy}{dt} - \alpha x = 0, \tag{1}$$

and

$$\frac{d^2x}{dt^2} + 2n\frac{dx}{dt} + \alpha^2 x = 0, \tag{2}$$

where α and n are constants. The integrals of (1) are well known to be

$$x = a\cos(\alpha t + \theta), \quad y = a\sin(\alpha t + \theta), \tag{3}$$

where a and θ are constants. The integral of (2) is well known to take different forms according as $n^2 > \alpha^2$, $n^2 = \alpha^2$, or $n^2 < \alpha^2$. Equation (2) represents the phenomena of damped vibrations. When $n = 0$, it represents simple harmonic motion in a straight line ; while equations (1) represent the rectangular components of uniform motion in a circle. The usual methods of integrating these equations require some knowledge of the theory of differential equations in general. It is the object of Professor Woodward's to show how to integrate them without the aid of such knowledge. To this end, observe that (1) and (2) are comprised in the following equations of the first order :

$$\frac{dx}{dt} + \alpha y + 2nx = 0, \quad \frac{dy}{dt} - \alpha x = 0. \tag{4}$$

These equations give

$$\frac{dx}{2nx + \alpha y} = -\frac{dy}{\alpha x} = -dt,$$

whence, by the introduction of two factors β, γ,

$$\frac{\beta dx}{2nx + \alpha y} = -\frac{\beta\gamma dy}{\alpha\gamma x} = -\beta dt,$$

or

$$\frac{\beta dx - \beta\gamma dy}{(2n + \alpha\gamma)x + \alpha y} = -\beta dt. \tag{5}$$

The first member of this equation will be an exact differential if
$$\beta = 2n + a\gamma, \quad \text{and} \quad -\beta\gamma = a, \tag{6}$$
whence
$$a\gamma = -n \pm \sqrt{n^2 - a^2}.$$
Thus the two values of β derived from (6) are
$$\beta_1 = n + \sqrt{n^2 - a^2}, \quad \beta_2 = n - \sqrt{n^2 - a^2}. \tag{7}$$
Writing $\omega^2 = n^2 - a^2$, it is seen that
$$\begin{aligned} &\beta_1 = n + \omega, \quad \beta_2 = n - \omega \qquad (n^2 > a^2)\,; \\ &\beta_1 = n, \qquad\quad\ \beta_2 = n \qquad\qquad (n^2 = a^2)\,; \\ &\beta_1 = n + i\omega, \quad \beta_2 = n - i\omega \qquad (n^2 < a^2). \end{aligned} \tag{8}$$

Introducing β_1 and β_2 in (4), and observing the second of (6), there result
$$\frac{\beta_1 dx + a dy}{\beta_1 x + ay} = -\beta_1 dt, \quad \frac{\beta_2 dx + a dy}{\beta_2 x + ay} = -\beta_2 dt.$$
Integrating these and calling the values of x and y for $t = 0$, x_0 and y_0 respectively,
$$\beta_1 x + ay = (\beta_1 x_0 + a y_0) e^{-\beta_1 t},$$
$$\beta_2 x + ay = (\beta_2 x_0 + a y_0) e^{-\beta_2 t},$$
whence by elimination
$$\begin{aligned} (\beta_2 - \beta_1)x &= (\beta_2 x_0 + a y_0)e^{-\beta_2 t} - (\beta_1 x_0 + a y_0) e^{-\beta_1 t}, \\ a(\beta_2 - \beta_1)y &= (\beta_1\beta_2 x_0 + a\beta_2 y_0)e^{-\beta_1 t} - (\beta_1\beta_2 x_0 + a\beta_1 y_0)e^{-\beta_2 t}. \end{aligned} \tag{9}$$
Substituting in these the first pair of values from (8), there result
$$\begin{aligned} x &= e^{-nt}\left(x_0 \cosh \omega t - \frac{n x_0 + a y_0}{\omega} \sinh \omega t\right), \\ y &= e^{-nt}\left(y_0 \cosh \omega t + \frac{a x_0 + n y_0}{\omega} \sinh \omega t\right), \end{aligned} \qquad (n^2 > a^2). \tag{10}$$

Similarly, using in (9) the last pair of values in (8), or replacing ω in (10) by $i\omega$,
$$\begin{aligned} x &= e^{-nt}\left(x_0 \cos \omega t - \frac{n x_0 + a y_0}{\omega} \sin \omega t\right), \\ y &= e^{-nt}\left(y_0 \cos \omega t + \frac{a x_0 + n y_0}{\omega} \sin \omega t\right), \end{aligned} \qquad (n^2 < a^2). \tag{11}$$

For the intermediate case in which $n^2 = a^2$, or $\omega = 0$, it is seen at once from (10) or (11) that

$$x = e^{-nt}\{x_0 - a(x_0 + y_0)t\},\qquad\qquad (n^2 = a^2). \qquad (12)$$
$$y = e^{-nt}\{y_0 + a(x_0 + y_0)t\},$$

For the special case in which $n = 0$, and hence $\omega = a$, or for the case of simple harmonic motion specified by (1), (11) give

$$x = x_0 \cos at - y_0 \sin at, \qquad y' = y_0 \cos at + x_0 \sin at,$$

which become identical with (3) by means of the substitution $x_0 = a \cos \theta$, $y_0 = a \sin \theta$.

The following is an abstract of Dr. Miller's paper: The $\varphi(g)$ numbers which are less than g and prime to g constitute a group with respect to multiplication if the smallest positive residues mod g are taken in place of the products. This group G_1 is the group of isomorphisms of the cyclical group of order g. When G_1 is cyclical its order g_1 must be $p^a(p-1)$, where p is some odd prime number. From this it is clear that g_1 must be even whenever G_1 is cyclical. In fact g_1 is always even. If $g_1 = 2^\beta$, G cannot have more than two independent generators of the same order except for the special case when $g = 24$. In this case G_1 is clearly of type (1, 1, 1).

Professor Lovett's first paper is in summary as follows: Speculations relative to n-dimensional space have followed several fairly well-defined trends: 1° A direct extension of the cartesian geometry, which extension is to be regarded as nothing more than a convenient form of phraseology; in this form n-dimensional space sprang forth from the minds of Grassmann, Cayley, Gauss, and Cauchy, and the idea was likely familiar to Euler and Lagrange. 2° The transformation of the ordinary visualizable spaces of two and three dimensions into manifoldnesses of higher or lower dimensions by introducing space elements other than the point or its dual element; for example, the line geometry of Plücker, the sphere geometry of Lie, the five-dimensional manifoldness of all conics in the plane as an auxiliary to Ball's theory of screws; this category becomes more concrete perhaps than any other. 3° The absolute geometry of space; here would appear the famous dissertation of Riemann, the well-known memoirs of Helmholtz, Klein, and

Lie, and the elaborate treatise of Veronese. 4° The extension of the methods and results of ordinary differential geometry and analysis situs to spaces of many dimensions; to this class belongs the works of Christoffel, Beltrami, Bianchi, Cesàro, and Ricci, and the quite recent contributions of Darboux and his followers. 5° The direct extention of the problems and concepts of metrical and projective geometry of ordinary space, as exemplified in the memoirs of Jordan, d'Ovidio, and Veronese. 6° The theory of birational correspondences between n-dimensional aggregates as studied by Noether, Kantor, and Brill. 7° The descriptive geometry of space of n dimensions as begun in the papers of Veronese, Stringham, Schlegel, and Segre. 8° The kinematics of higher spaces as developed by Jordan, Clifford, and Beltrami. 9° The interpretation of n-dimensional geometry in the light of the theory of groups as exhibited by Klein, Lie, and Poincaré.

It is proposed in the present paper to make an expository contribution to the ninth and fourth of the above categories, constructing ordinary four-dimensional space by the method of Lie's theory of continuous groups and studying curves of triple curvature by the intrinsic analysis developed by Cesàro in his Lezioni di geometria intrinseca.

In Professor Lovett's second paper, which will appear in the *Annals of Mathematics*, the well-known property $(U, V) \equiv 0$ of the Jacobian system

$$Uf \equiv \frac{\partial f}{\partial x} + \zeta_1 \frac{\partial f}{\partial z} = 0, \qquad Vf \equiv \frac{\partial f}{\partial y} + \zeta_2 \frac{\partial f}{\partial z} = 0,$$

is employed to derive the ordinary criterion for the integrability of the equation

$$Pdx + Qdy + Rdz = 0.$$

The result extends itself immediately to the case of n variables.

Professor Hinton's paper investigates the principle of duality as it would exist in a fictitious geometrical world in which the distance of a point is measured by means of the difference, instead of the sum, of the squares of its abscissa and ordinate.

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