The functions y_α become here

$$y_0 = \sum_{h_2} \vartheta_2^{h_2} x_0 = x_0 + x_2 ; \quad y_1 = \sum_{h_1} \vartheta_1^{h_1} x_0 = x_1 + x_3 .$$

Now both y_0 and y_1 are zero ; they are thus not cyclic, and the rule breaks down.

Finally we observe that the treatment in Chapters X. and XII. of Kronecker's problem, of finding the necessary and sufficient form of the roots of all algebraically solvable equations of prime degree n, is far too condensed for so abtruse a matter. It is also lacking in rigor in two essential points. The question whether the functions

$$\psi_\mu = \sum_{r=0}^{n-1} \omega^{\mu r} x_r \qquad (\omega^n = 1)$$

vanish or whether the functions

$$y_q = \psi_{g^q} \psi_{g^{q-1}}^{-g} \qquad (q = 1, 2, \cdots, n-1)$$

are distinct is not discussed.

Before closing we beg to have it clearly understood that our criticisms have been made on the supposition that the volume in hand is to serve as an introduction to the modern theory of the algebraic solution of equations. To one who is already familiar with the elements of this theory, the present work will give much interesting and valuable information, particularly in regard to the methods peculiar to Kronecker. It may then serve in some measure as a preparation toward studying the papers of this great master.

JAMES PIERPONT.

YALE UNIVERSITY,
March, 1900.

ELEMENTS OF THE CALCULUS.

The Elements of the Differential and Integral Calculus, based on the Kurzgefasstes Lehrbuch der Differential- und Integralrechnung, von W. Nernst und A. Schönflies. By J. W. A. YOUNG and C. E. LINEBARGER. New York, D. Appleton and Co., 1900. 8vo., xvii + 410 pp.

OF the various new text-books on the calculus, this recent joint publication by a teacher of mathematics and a teacher of physics and chemistry will doubtless attract much interest, based as it is upon the German work, intended primarily for chemists, which appeared in 1896 from the pens of Professor Nernst, the celebrated exponent of the new physical chemistry, and Professor Schönflies, widely known for his investigations in pure and applied mathematics. Having taught calculus to classes of technical students and having been formerly a chemist on a state geological survey, the reviewer finds himself fully in sympathy with such a work and ready to approve even a greater number of examples and illustrations from the fields of chemistry and physics.

To the students of mathematics, astronomy, advanced physics, or technology the book is intended as a "pre-view" or general survey of the field, giving a clear insight into the fundamental principles, methods, and results, rather than the more elaborate and complex parts of the subject. The book is also offered to the general student in the belief that an elementary knowledge of the calculus should round out the mathematical career of the students seeking liberal culture. In the opinion of the reviewer the work is well adapted to those teachers or students of the natural or exact sciences who would privately acquire through an interesting and rigorous text a fair knowledge of the calculus and a view of its wide application in all domains of science.

The book is in two senses an introduction to the subject. In the first place, it presupposes only a knowledge of geometry, plane trigonometry, and the very simplest parts of algebra. The first seventy-five pages are devoted to an introduction to analytic geometry, giving as much as is needed in the later chapters. Chapters on higher algebra are inserted when needed for their applications to the calculus. Twenty pages are devoted to limits, in which the pedagogical turn of mind of the author or authors is clearly discernible. Several definitions of limits are given, graded in difficulty up to the modern treatment by means of the standard "epsilon." Seven pages on logarithms, ten pages of illustrations of partial fractions, fifty pages on infinite series, of which fifteen are devoted to the elementary algebraic theory, are given at convenient places in the text.

In the second place, the elementary character of the book may be indicated by the remark that the authors avoid the subjects curvature and elliptic integrals, so that the problem of the rectification of the ellipse and the problem of the simple pendulum are not even mentioned.

The elementary parts of the calculus are treated with admirable clearness and pedagogical insight, and yet the proofs possess all the rigor demanded by the modern methods in analysis. Each fundamental process is first illustrated by simple examples chosen from geometry, mechanics, physics, or chemistry, and afterwards the general method is exhibited in mathematical formulæ.

The method of limits is used exclusively. In view of the strict avoidance of differentials, it would seem more pedagogical to replace the term differentiation by derivation. The earliest introduction (on p. 108) of the symbol dy/dx for the derivative might have been done with more care, giving temporarily the notation $\frac{d}{dx}(y)$ or the alternative notation $D_x y$. The reader would be spared the confusion usually attendant upon the first notation. That the authors themselves will admit the justness of this criticism is apparent from their remark on p. 172, "In the Differential Calculus we grew accustomed not to keep $\frac{d}{dx}$, the symbol for the operation of differentiation, invariably separated sharply from the quantity to which the operation was to be applied, but sometimes to write for compactness $\frac{dy}{dx}$, ⋯ instead of $\frac{d}{dx}y$, ⋯." As a matter of fact the former symbol occurs six times before the latter is (on p. 116) first introduced!

It is not easy to make students *feel* that dy/dx is not a fraction. Furthermore, of the two equivalent formulæ,

$$\frac{dy}{du}\cdot\frac{du}{dx}=\frac{dy}{dx}, \quad \frac{d}{du}(y)\cdot\frac{d}{dx}(u)=\frac{d}{dx}(y),$$

the necessity of formal proof is not usually felt by the student in the case of the first form, but is apparent in the case of the second. Instead of the usual short proof of this formula (based upon the lemma that the limit of a product equals the product of the limits of the factors), the authors devote pp. 150–152 to a rather complicated proof. While the reviewer is accustomed to give both, he would rather omit the latter than the former.

The authors never define dx and dy as separately existing quantities,* but still retain the historic symbol $\int f(x)dx$ for

* This may be done with rigor by taking them to be *any* two quantities (infinitesimal or not) whose ratio is the derivative. For application in integral calculus, we may also define dx as an arbitrary infinitesimal increment of x and dy as that dependent function equal to the product of dx and the derivative y'. Geometrically, dx and Δy refer to the curve, while dx and dy refer to its tangent.

the function whose derivative with respect to x is $f(x)$. The step involved is certainly logical, but rather severe logic. Here again the (temporary) use of an alternative symbol, as $D_x^{-1} f(x)$, coupled with a reference to the definite integral, would bring this heroic adherence to the method of limits somewhat nearer to the plane of the student. Indeed, what student will not stumble at the statement on p. 252: "The sign $\int$ represents a form of *s* now obsolete, standing for the word *sum*, and *ydx* represents the type of the terms of the sum." The refusal to define *dx* makes the formula (p. 192) for integration by parts quite cumbrous. A re-adjustment must certainly take place somewhere in a student's career, if he is to continue his mathematical or physical studies in the modern literature.

On p. 296 and in the examples on p. 297 and pp. 301–307, it is required to find dy/dx when y is given as an implicit function of x, say $\varphi(x, y) = 0$. It would seem desirable to explain the usual method of taking the derivative with respect to x of both members of the equation and then solving for dy/dx. This procedure is more practical (even if it involves the calculation of the same derivatives) since it does not require the retention of a formula somewhat difficult to remember.

On p. 300, formula (5), δ is used four times for ∂.

The last chapter, a literal translation of the corresponding chapter of the German text, is highly interesting and useful. It gives a new and successful method for the differentiation or integration of functions found empirically, *i. e.*, given by a table of experimental results. In it occurs (p. 390) a confusing change of notation. It would make for clearness to omit the x introduced, but not used, in the regular text, and to explain, elsewhere than in the example given in small print, the p and θ actually used.

By this time it will be seen that the reviewer has sought in vain for very serious grounds for criticism of the text. He may therefore be permitted to congratulate the authors upon the skill with which they have completed such an interesting and valuable introduction to the calculus.

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