THESES IN MATHEMATICS AT THE UNIVERSITY OF PARIS.

1. *Sur quelques points de la théorie des fonctions*, par M. L. Desaint. Paris, Gauthier-Villars, 1897. 4to, 74 pp.
2. *Sur une classe particulière de groupes hyperabéliens*, par M. Henry Bourget. Ibid., 1898. 4to, 90 pp.
3. *Sur l'intégration des équations de la chaleur*, par M. Édouard LeRoy. Ibid., 1898. 4to, 258 pp.
4. *Les équations différentielles linéaires et la théorie des groupes*, par M. F. Marotte. Ibid., 1898. 4to, 92 pp.
5. *Essai sur une théorie générale de l'intégration et sur la classification des transcendantes*, par M. Jules Drach. Ibid., 1898. 4to, 144 pp.*

These five theses were presented to the faculty of sciences of the University of Paris for the degree of doctor in mathematical science; they were sustained in order of time as placed above during the academic year from November, 1897, to June, 1898, inclusive.

Before proceeding to a brief review of each it may be interesting to observe that the creation of the faculties of mathematical and physical science in France dates from March 17, 1808.† The first degree of this kind conferred by the faculty of sciences at Paris was given to Pierre-Louise-Marie Bourdon in 1811; his two theses maintained respectively on the 19th and 23d of March of that year were 1° the motion of a solid body solicited by any accelerative forces and turning about a fixed point, and 2° the theory of the elliptic movement of the planets according to the principle of universal gravitation and its application to the determination of the masses of some planets.

From 1811 to 1890 inclusive there were conferred one hundred and eighty-four such degrees at Paris, and forty-four by the other universities of France.‡ The average age of the candidates is thirty years and a fraction; the majority lie in the range from twenty-three to thirty-four years, the average for each of these years being twelve; there were thirteen at twenty-three, eighteen (the maximum) at twenty-

* The memoirs of Desaint, LeRoy, and Drach are also published in the *Annales de l'Ec. norm. sup.*, for the respective years 1897, 1897–8, 1898; those of Bourget and Marotte appeared in the *Annales de la Faculté des Sciences de Toulouse* for 1898.

† See Maire's Catalogue des thèses de sciences soutenues in France de 1810 à 1890 inclusivement, Paris, H. Welter, 1892.

‡ For the same period the respective numbers in the physical sciences were 281 and 61; in the natural sciences, 236 and 67.

eight, and ten at thirty-four ; the ages range from seventeen years to fifty-five ; Joseph Bertrand was the youngest candidate ; the next age is twenty, and with but four exceptions, every age from twenty to fifty-five is represented ; five candidates attained the degree at twenty-one and it was conferred on five who had passed the fifty-year mark ; the average age of the first ten of this period (1810–1890) was twenty-six years, that of the last ten is also twenty-six, the fraction being slightly larger for the former.

1. In the first part of his memoir Desaint studies the distribution of the zeros of a function. This problem is one way of approaching the general problem of the resolution of equations ; he does not propose to solve the very difficult problem of determining their roots point by point, but to limit the regions of the plane of the complex variable within which a function can vanish. The case of a polynomial with real roots is one of the rare examples in which the zeros of a function can be found point by point ; if certain roots of a polynomial are commensurable these are obtained immediately ; as to incommensurable roots the author proceeds thus : the value of the root sought is placed between two commensurable numbers, one, a, greater, and the other, b, less ; consider the segment ab of the axis of reals and to each zero make correspond an analogous segment ; the ensemble E of these segments represents the precision with which the equation has been resolved ; from the author's point of view E limits the region of the plane where the roots of the polynomial are found.

The first chapter presents the geometric method of which Desaint makes use ; it rests upon the following simple principle : consider an ensemble of segments F starting from a point z ; if all these segments are situated below a straight line D, their resultant is essentially different from zero and situated below D. Applying this to rational fractions the author is led to a fundamental theorem which he applies to the roots of a polynomial, to algebraic functions, and to the zeros of uniform functions with polar discontinuities. Certain propositions found complete the investigations of Lucas, Berloty, and Cesàro.

Desaint is occupied in the second chapter with definite integrals studied by Hermite. Theorems on uniform functions are arrived at relative to the distribution of the values of the variable which make these functions assume a given value u. A theorem is presented which is applied to elliptic and hyperelliptic integrals, and especially to hypergeometric integrals.

In the second part of the memoir the theorem of Cauchy is used to turn the preceding theorems to a solution of the problem of the distribution of the values of the variable which assign a given value to a uniform function. The problem of finding the value of a function at any point when its values on a contour are given has received a great deal of attention; the author proposes the inver e problem of finding the values of the variable which assign the value u to a function, when the function is given by its values along a contour. He calls particular attention to this proposition: Let $F(z)$ be a uniform function for which infinity is an ordinary point; trace a circle C enclosing all the discontinuities of $F(z)$ and as near as we wish to the convex contour of a minimal surface enclosing these points; designate by M the maximum modulus of $F(z)$ upon the circle C of radius R; the values of z for which $F(z)$ takes the value u are on the interior of the circle T concentric with C and having as radius $R\sqrt{2}\left(1+\frac{M}{|A-u|}\right)$. From this theorem we deduce with facility a general theorem on the continuity of functions. The second part closes with a study of integral functions and of integrals of differential equations.

M. Desaint's second thesis consisted of general theorems of Galois upon algebraic equations; application to resolution by radicals of equations of first degree and to the construction of regular polygons.

2. M. Bourget, at the suggestion of Professor Picard, studies the particular hyperabelian group pointed out by the latter geometer in his memoir in the first volume of the fourth series of *Liouville's Journal* and previously in the *Comptes Rendus* for 17 March, 1884.

This particular group has a double origin. We can derive it from the transformation of the first order of the abelian functions of the second kind;* or it can be considered as isomorphic to the group of transformations similar arithmetically of the quaternary form

$$u_1^2 - Du_2^2 + u_3 u_4.$$

The thesis falls into three parts. In the first of these, after generalities on similar substitutions of quaternary quadratic forms and on the arithmetic study of such forms, Bourget shows how the group which he studies is situated relative to the analogous groups considered by Goursat and Bianchi.

* Hermite, "Sur la transformation des fonctions abéliennes," *Comptes rendus*, vol. 40 (1855).

The second part is devoted to the study of the group. Seeking first the explicit form of the substitutions, the discontinuity for the imaginary values $\xi = \xi_1 + i\xi_2$, $\eta = \eta_1 + i\eta_2$ appertaining to the domain δ ($\xi_2 > 0$, $\eta_2 > 0$) is then demonstrated; the substitutions of the group are reduced to five of them, which fundamental substitutions present a remarkable analogy to the fundamental substitutions of the modular group; the author also brings to light an infinite number of subgroups analogous to the congruence subgroups of the modular group.

Bourget inquires in the third part how the ten functions

$$\vartheta(0,0/\tau_{11}, \tau_{12}, \tau_{22})$$

even with zero arguments, are affected by the transformations of the group, and from this study deduces a process which, by the aid of the theta functions, constructs an infinitude of functions which are reproduced by all the substitutions of the group. The moduli of Borchardt and Richelot are invariant under certain subgroups. In conclusion the general properties, according to Picard, of the fonctions of the group are given, and it is shown that, notwithstanding the cusp situated on the limit of the domain δ (whose existence was established in the second part), any three of these functions are connected by an algebraic relation.

M. Bourget's second thesis was the exposition of the method of Gauss for the calculation of secular variations.

3. The object of LeRoy's memoir is the resolution of certain problems of the integral calculus which support the mathematical theory of heat. These problems are referred to certain partial differential equations, of which Laplace's equation is the type. Each of them consists in establishing a principle analogous to that of Dirichlet.

The author proposes 1° to construct an integral $V(x, y, z)$ of the equation

$$\Delta V + aV_x + bV_y + cV_z = f(x, y, z, V) + \varphi(x, y, z),$$

which shall be definite and continuous in every point of a closed domain, on the boundary of which the function sought will be constrained to take values given in advance; 2° to design an integral $V(x, y, z, t)$ of the more general equation which he calls the equation of Fourier

$$\begin{aligned}\Delta V + a(x, y, z)V_x + b(x, y, z)V_y + c(x, y, z)V_z \\ = f(x, y, z, V) + \varphi(x, y, z, t) + \psi(x, y, z)V_t,\end{aligned}$$

where $adx + bdy + cdz$ is an exact integral, which integral shall be definite and continuous for every point of a closed domain and for every positive value of the time, reducing for $t = 0$ to a function (x, y, z) given in advance and taking at the surface of the body values assigned *a priori*.

The plan followed in the paper may be briefly sketched thus: A remark made by M. Paraf* permits the author to extend the principal propositions relative to Laplace's equation to equations of a general type whatever be the number of variables; then having reduced the method called by Poincaré the méthode du balayage to a canonical form, he deduces by a uniform process the complete resolution of the problems of the theory of heat, at least so far as existence theorems are concerned.

The equations of thermic equilibrium are considered from the point of view of the generalization of Dirichlet's principle; a point of view taken by Picard in several well-known memoirs.† The calculations are made in three point variables. The méthode du balayage employed is independent of any preliminary theory of Laplace's equation; for linear equations the author demonstrates the principles of Dirichlet and its generalization at one stroke applicable alike to plane and space; as to non-linear equations a new method is proposed, which succeeds for all such equations as are suggested by the theory of heat.

In the second part the author studies the properties of functions which he calls the fundamental harmonic functions attached to a closed surface; his guide here is a memoir of Poincaré.‡ The existence theorems demonstrated lead to developments in series by which, after Lamé, the construction of the solution of Dirichlet's problem would be sought. The author is limited here to Laplace's equation.

He arrives finally at the equations of cooling of solid bodies and to Fourier's problem. For the case of a homogeneous body an imitation of the méthode du balayage is employed. The author shows that it is again possible to find, *a posteriori*, solutions under the form of series to which he gives the name of Lamé; the process throws new light on several questions of physics; for example, the problem of vibrating membranes. In conclusion LeRoy considers the possibility of applying to the general equations of a variable régime the process imagined by Picard for the equations of permanent régime.

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* A. Paraf, "Sur le problème de Dirichlet et son extension au cas de l'équation linéaire générale," *Toulouse Annals*, vol. 6 (1892).

† Picard, *Acta Math.*, vol. 12; *Liouville's Journal*, 1890, 1896; *Jour. de l'Éc. poly.*, cahier 60.

‡ Poincaré, *Rend. del circ. mat. di Palermo*, 1894; *Acta Math.*, 1896.

M. LeRoy's second thesis was on the integrals of total differentials and on the double integrals of the first kind in the theory of algebraic surfaces.

4. Marotte's memoir has to do with the question of extending to linear differential equations the ideas introduced by Galois into the theory of algebraic equations. The point of view was discovered by Picard* in the following theorem : To every nth order linear differential equation there corresponds an algebraic group of linear transformations in n variables, which possesses properties analogous to those of the Galois group of an algebraic equation.

Vessiot † demonstrated completely the double property of this group of rationality and made evident its close relation to the problem of integration. Drach ‡ extended the theory of Galois to partial differential equations of the first order.

The first part of Marotte's paper is occupied with an analytical study of the singularities of linear differential equations having rational coefficients and the classification of the transcendents which integrate them. The author shows that to each singular point a of such an equation there is attached a group of linear transformations which plays the same rôle in the study of a singularity as does the group of Galois in the resolution of an algebraic equation, or the group of rationality in the integration of a linear differential equation. This group is called the group of meromorphism, because its differential invariants are expressed by functions of x meromorphic at point a ; the group characterizes the nature of the singular point. For an nth order equation there are as many classes of singular points as there are subgroups in the linear group of n variables.

After having established the relations which exist between the group of rationality, the group of monodromism and the groups of meromorphism, the author enlarges the field of application of the methods of Galois and shows how they lead to the notion of group of monodromism. The group of rationality gives the position of the integrals with respect to the ensemble of rational functions exactly as the group of monodromism gives their position relative to the ensemble of uniform functions. The preceding results are then applied to the classification of the transcendents which integrate linear differential equations with rational coefficients ; the notion species of Riemann is the ultimate classifying element. A method is presented by which it can be recognized when two given equations are of the same species.

* *Comptes rendus*, 1883 ; *Toulouse Annals*, 1887.

† *Ann. de l'Ec. norm.*, 1892.

‡ *Comptes rendus*, 1893, 1895.

The second part of the memoir is devoted to the determination of the group of rationality of a differential equation with rational (or algebraic) coefficients. A new classification of homogeneous linear groups enables the author to refer the determination of the group of rationality to the resolution of the problem of determining whether a linear differential equation with rational coefficients admits of an integral whose logarithmic derivative is rational or algebraic; hence the conclusion that we can always determine the group of rationality of a linear equation of order two, three or four, or refer the determination to the study of an abelian integral. The author thus finds all the possible cases of reduction of a linear equation of the fourth order. He refers the integration problem to its canonic form. The method used extends itself immediately to equations of higher order.

In the concluding chapter the work of von Koch * is applied to the study of the question of finding whether a linear equation with rational coefficients admits of an integral whose logarithmic derivative is meromorphic (normal integral), to which problem Marotte refers the determination of the group of monodromism attached to a singular point of a linear equation.

M. Marotte's second thesis was on the general principles of dynamics.

5. The remarkable thesis of Drach calls for more extended notice than can be given to it in the space at command for this article. A suitable review of the memoir will appear in a subsequent number of the BULLETIN.

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NOTES.

AT the annual meeting of the Society, December 28, 1899, President R. S. WOODWARD will deliver a presidential address on "The century's progress in applied mathematics."

THE president (Lord KELVIN), the vice-presidents and the secretaries of the London mathematical society have been renominated to serve in the same capacity on the council for the ensuing year. Professor W. BURNSIDE,

* *Acta Math.*, vol. 18; *Comptes rendus*, 1893.