ON A DEFINITIVE PROPERTY OF THE COVARIANT.

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THE general homogeneous entire polynomial of degree n in k variables x may be denoted by

$$F_n(x_1, x_2, \cdots, x_k) \equiv \sum c_{e_1 e_2 \ldots e_k} x_1^{e_1} x_2^{e_2} \cdots x_k^{e_k}$$

where $e_1 + e_2 + \cdots + e_k = n$. Let

$$\varphi_n(\xi_1, \xi_2, \cdots, \xi_k) \equiv \sum \gamma_{e_1 e_2 \ldots e_k} \xi_1^{e_1} \xi_2^{e_2} \cdots \xi_k^{e_k}$$

represent the polynomial into which F is converted by the substitutions

$$\begin{aligned} x_1 &= \lambda_{11}\xi_1 + \lambda_{12}\xi_2 + \cdots + \lambda_{1k}\xi_k, \\ x_2 &= \lambda_{21}\xi_1 + \lambda_{22}\xi_2 + \cdots + \lambda_{2k}\xi_k, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ x_k &= \lambda_{k1}\xi_1 + \lambda_{k2}\xi_2 + \cdots + \lambda_{kk}\xi_k, \end{aligned}$$

where the λ's are subject to a single restriction : their determinant D shall not assume the value zero.

If there is such an entire homogeneous polynomial

$$\psi_m(x_1, x_2, \cdots, x_k) \equiv \sum h_{e_1 e_2 \ldots e_k} x_1^{e_1} x_2^{e_2} \cdots x_k^{e_k},$$

where $e_1 + e_2 + \cdots + e_k = m$ and where each coefficient h is an entire homogeneous polynomial of degree p in the coefficients c of F, that

$$\psi_m(\xi_1, \xi_2, \cdots, \xi_k) \equiv M \cdot \psi_m(x_1, x_2, \cdots x_k),$$

the γ's entering the left member of the identity as the c's enter ψ_m of the right member, then $\psi_m(x_1, x_2, \cdots, x_k)$ is named covariant or invariant of F according as $m > 0$ or $= 0$.

Supposing such a function ψ to exist, it remains to determine the nature of the factor M. The ξ's and the γ's being linear respectively in the x's and the c's, the two members of the identity in question are, apart from the factor M, each of degree m in the x's and of degree p in the c's. It follows that M is a function of the λ's only. M is, moreover, rational since ψ is an entire polynomial and the equations of transformation are linear.

We may, then, write $M \equiv P_1 : P_2$, where the P's are homogeneous entire polynomials in the λ's and contain no common factor. Suppose first that M is not identically equal to a constant k'. If P_1 be not of the form $k_1 D^\rho$, then the λ's may be so chosen as to reduce P_1 to zero without causing either P_2 or D to vanish; for, if not, the locus $P_1 = 0$ would be a component of the locus $P_2 \cdot D = 0$, which is impossible inasmuch as D is not factorable and P_1 and P_2 have no factor in common. But if the λ's be chosen as indicated, $M = 0$ and consequently $\psi_m(\xi_1, \xi_2, \cdots, \xi_k)$ is seen to be identically zero, a result incompatible with the original supposition that the c's are entirely arbitrary. In like manner, if P_2 is not of the form $k_2 D^\rho$, it is possible by a suitable choice of the λ's to cause P_2 to vanish without reducing either P_1 or D to zero; but under this hypothesis covariants could not exist, for, on multiplying by P_2, the left member and hence the right member of the identity would be identically zero. Finally, if $M = k'$, then we may write $M = k' D^0$. Accordingly M must be of the form $k'' \cdot D^\rho$. By means of the transformation $x_1 = \xi_1$, $x_2 = \xi_2$, $\cdots$, $x_k = \xi_k$, it is readily found that $k'' = 1$. It thus appears that, under the definition, either no covariant exists or the factor M is of the form D^ρ.

In vol. I. of Jordan's Cours d'Analyse is found a proof of this proposition, in which the argument turns on the reversibility of the substitutions. A second demonstration, by Professor E. B. Elliott, occurs in vol. 16 of the *Messenger of Mathematics* and in his Introduction to the algebra of quantics. Here M is shown to be homogeneous, its degree in the λ's is determined, and then by help of the reversibility of the transformation a partial differential equation connecting M with its derivatives with respect to the λ's is obtained, whence the form of M is readily ascertained. In still a third proof by Professor Thomas S. Fiske, in vol. 19 of the journal just cited, the proposition defining the form of M is derived as a corollary from the converse there established of the multiplication theorem for determinants: A rational entire function having n^2 arguments and subject to the same law of multiplication as a determinant is a power of a determinant.

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