THE APRIL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

A REGULAR meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York City on Saturday, April 29, 1899. Twenty-eight persons were in attendance, including the following twenty-four members of the Society :

Professor Maxime Bôcher, Dr. A. S. Chessin, Professor F. N. Cole, Dr. W. S. Dennett, Professor T. S. Fiske, Mr. G. B. Germann, Dr. A. A. Himowich, Dr. J. I. Hutchinson, Professor Harold Jacoby, Mr. C. J. Keyser, Professor Pomeroy Ladue, Dr. Emory McClintock, Mr. James Maclay, Dr. D. A. Murray, Professor W. F. Osgood, Mr. J. C. Pfister, Professor James Pierpont, Professor J. K. Rees, Professor C. A. Scott, Dr. W. M. Strong, Professor E. B. Van Vleck, Professor J. M. Van Vleck, Miss E. C. Williams, and Professor R. S. Woodward.

The President of the Society, Professor R. S. Woodward, occupied the chair during the morning and afternoon sessions. The Council announced the election of the following persons to membership in the Society : Professor Rosser D. Bohannan, Ohio State University, Columbus, Ohio ; Professor Florian Cajori, Colorado College, Colorado Springs, Colo.; Mr. John S. Morris, Central High School, Philadelphia, Pa.; Dr. John V. E. Westfall, Cornell University, Ithaca, N. Y. Eleven applications for membership were reported. Professor Thomas F. Holgate was appointed Acting Secretary for the Summer Meeting.

The Council announced through the President that, the requisite financial guarantees having already been secured, it was prepared to recommend that the publication of the *Transactions* of the Society be begun in January, 1900. The By-Laws were then amended, in accordance with the recommendation of the Council, so as to define the status of the *Transactions* and to provide for the creation of its editorial board. This board consists of three editors appointed by the Council. As originally organized, it is composed of Professor Eliakim Hastings Moore, to serve until February, 1904 ; Professor Ernest W. Brown, to serve until February, 1903 ; and Professor Thomas Scott Fiske, to serve until February, 1902.

The following papers were presented at the meeting :

(1) Dr. J. I. HUTCHINSON : "The asymptotic lines of the Kummer surface."

(2) Dr. L. E. DICKSON: "The known finite simple groups."

(3) Mr. E. B. WILSON: "Note on the functions satisfying the functional relation $\varphi(x)\varphi(y) = \varphi(x+y)$."

(4) Dr. A. S. CHESSIN: "On the differential equations of dynamics."

(5) Professor CHARLOTTE ANGAS SCOTT: "A proof of Noether's fundamental theorem."

(6) Dr. G. P. STARKWEATHER: "Non-quaternion systems containing no skew units."

(7) Professor É. GOURSAT: "Sur la définition générale des fonctions analytiques d'aprés Cauchy."

(8) Professor F. MORLEY: The value of

$$\int_0^{\pi/2} (\log 2 \cos \varphi)^m \varphi^n d\varphi."$$

(9) Professor E. W. BROWN: "An elementary illustration of the connection between the current and the height of the water in a tidal estuary."

(10) Dr. W. M. STRONG: "The determination of non-quaternion number systems in six units."

(11) Professor E. O. LOVETT: "Curves of multiple curvature."

(12) Professor JAMES PIERPONT: "Elliptic functions."

(13) Mr. C. J. KEYSER: "On a definitive property of the covariant."

In the absence of the authors the papers of Professor Morley, Professor Lovett, and Mr. Keyser were read by title; Dr. Dickson's paper was read by the Secretary, Mr. Wilson's and Professor Goursat's by Professor Osgood, and Professor Brown's by Professor Woodward. The papers of Mr. Wilson and Professor Goursat were offered to the Society through Professor Osgood, Dr. Starkweather's through Professor Pierpont. The papers of Dr. Hutchinson, Dr. Dickson, Professor Pierpont, and Mr. Keyser will appear in later numbers of the BULLETIN. Abstracts of the other papers are given below.

Mr. Wilson showed that if a function exists which is defined for all values of the argument, is single valued, satisfies the functional relation $\varphi(x)\varphi(y) = \varphi(x+y)$, and has a discontinuity at a single point, then, in the neighborhood of any value x_0 of the argument whatsoever, the function takes on values arbitrarily near to any preassigned positive value y_0. This paper will be published in the *Annals of Mathematics*.

Professor Chessin's paper is in abstract as follows: When the vincula (liaisons) of a moving system contain t explicitly in their equations, we may integrate the differential equations of motion as follows. Let these equations be given in canonic form

$$(1) \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial q}; \qquad \frac{dq}{dt} = \frac{\partial H}{\partial p}.$$

Then we may put

$$H = H_0 + \Omega$$

where H_0 denotes the Hamiltonian function H with the assumption that $t =$ const. in the equations of the vincula. We therefore first integrate the system

$$(2) \qquad \frac{dp}{dt} = -\frac{\partial H_0}{\partial q}; \qquad \frac{dq}{dt} = \frac{\partial H_0}{\partial q},$$

and then, as in the theory of perturbations, pass to the integrals of (1) by integrating the canonic system

$$(3) \qquad \frac{d\alpha}{dt} = \frac{\partial \Omega}{\partial \beta}; \qquad \frac{d\beta}{dt} = -\frac{\partial \Omega}{\partial \alpha},$$

the α, β being the constants of integration of system (2).

If it is possible to eliminate t from the equations of the vincula by introducing a moving system of coördinates $(O.\Xi YZ)$ instead of the fixed system (XYZ), then Ω may be expressed as follows:

$$\Omega = -L - \left(\frac{dN}{dt} - \frac{\partial N}{\partial t}\right) - \tfrac{1}{2}\Sigma\Sigma\Phi_{\lambda\mu}\frac{\partial \varphi_\lambda}{\partial t}\frac{\partial \varphi_\mu}{\partial t}$$

where $\frac{\partial}{\partial t}$ means partial differentiation with regard to t, while the other letters denote respectively:

$$L = \omega M_0 \cos(\omega M_0),$$

ω being the angular velocity of rotation for the system $(O.\Xi YZ)$ and M_0 the principal moment about O of the momentum of relative motion of the given material system (*i. e.*, relatively to the system ΞYZ);

$$N = m w_0 \rho_c \cos(w_0, \rho_c),$$

m being the total mass of the given system, w_0 the velocity of the point O, ρ_c the radius vector of the center of inertia (from O);

$$\Phi_{\lambda\mu} = \frac{\partial lg\Phi}{\partial \varphi_{\lambda\mu}}$$

$$\Phi = | \varphi_{\lambda\mu} |$$

$$\varphi_{\lambda\mu} = \Sigma \frac{1}{m}\left(\frac{\partial \varphi_\lambda}{\partial x_i} \frac{\partial \varphi_\mu}{\partial x_i} + \frac{\partial \varphi_\lambda}{\partial y_i} \frac{\partial \varphi_\mu}{\partial y_i} + \frac{\partial \varphi_\lambda}{\partial z_i} \frac{\partial \varphi_\mu}{\partial z_i}\right);$$

and, finally,

$$\varphi_\lambda(x_1, y_1, z_1, \cdots, x_i, y_i, z_i, \cdots, t) = 0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

being the equations of the vincula.

For a solid body a similar expression is obtained for Ω, the difference being only in the expressions $\varphi_{\lambda\mu}$.

The method here discussed is an application of the author's treatment of relative motion in general, as mentioned in two previous papers read before this Society (August 19 and October 29, 1898).

The proofs hitherto given of Noether's fundamental theorem are purely algebraic, befitting the algebraic use of the theorem. But it is also of primary importance in geometry, hence it seems worth while giving a proof that depends on geometrical ideas, not on elimination and convergent expansions. Professor Scott's paper presents such a proof. Let curves U, V intersect at points of multiplicity q_1r_1, q_2r_2, $\cdots$ on the two curves, and consider a curve through these points, with multiplicity still to be determined. Denote any curve that has this multiplicity by Ω. It is shown that if every Ω of a given order N, *i. e.*, every Ω_N, is of the form $BU + AV$, then this holds also for the next lower order, *i. e.*, $\Omega_{N-1} \equiv B_1U + A_1V$, hence for every Ω of order $< N$. Now the conditions imposed by the points are certainly independent if N be sufficiently great; and in this case the expressibility of Ω in the form $BU + AV$ requires that the multiplicity at the intersections be $\geqq q + r - 1$. Thus Noether's theorem is proved for the case that is of most general applicability in geometrical investigations. This paper will appear in the *Mathematische Annalen.*

Dr. Starkweather discussed a special class of non-quaternion number systems. It can be assumed that the given system is irreducible, otherwise it falls into irreducible systems of the type considered. We can take for one unit the modulus, η. Scheffers* has shown that, the system being of order u, we can take the remaining units $u_1, u_2, \cdots, u_{n-1}$, such that $u_i u_j$ and $u_j u_i$ $(j \leqq i)$ are linear in $u_1, u_2, \cdots, u_{j-1}$, and that any number expressible in these u's, say v, satisfies the equation $v^\lambda = 0$ where $\lambda \leqq n$.

Let $\lambda = n - \delta$. δ is called the deficiency of the system. Then a number σ must exist such that $\sigma^{n-\delta-1} \neq 0$. The quantities $\sigma, \sigma^2, \cdots, \sigma^{n-\delta-1}$ are linearly independent, and can be used as units $w_1 = \sigma^{n-\delta-1}$, $w_2 = \sigma^{n-\delta-2}$, $\cdots$, $w_k = \sigma^{n=\delta-k}$, $w_{n-\delta-1} = \sigma$. The multiplication table for the w's follows at once. There must be δ more units aside from η. It is possible to choose for these certain of the u's, $u_{z_1}, u_{z_2}, \cdots, u_{z_\delta}$, where $z_1 < z_2 \cdots < z_\delta$, such that $u_1, \cdots, u_{z_1-1}$ are expressible in the w's, and $u_{z_\alpha+1}, \cdots, u_{z_{\alpha+1}-1}$ are expressible in the w's and $u_{z_1}, u_{z_2}, \cdots, u_{z_\alpha}$. By replacing u_{z_α} $(\alpha = 1, 2, \cdots, \delta)$ by

$$\tau_\alpha = u_{z_\alpha} + a_{\alpha,\,2}\, w_2 + a_{\alpha,\,3}\, w_3 + \cdots + a_{\alpha,\,(n-\delta-1)}\, w_{n-\delta-1}$$

it is possible to have the table assume the following form: $\tau_\alpha w_{n-\delta-k}$ and $w_{n-\delta-k}\tau_\alpha$ are zero if $k > \alpha$; contain only w_1 if $k = \alpha$; and if $k < \alpha$ they are linear functions of

$$w_1, \cdots, w_{\alpha-k+1};\ \tau_1, \cdots, \tau_{\alpha-k}.$$

$\tau_\alpha \tau_\beta (\beta \leqq \alpha)$ is a linear function of $w_1, \cdots, w_\alpha$; $\tau_1, \cdots, \tau_{\alpha-1}$. τ_δ occurs only in its column and row, in the product with η. If we remove this column and row we have a system in $n - 1$ units. Since σ is not removed the equation satisfied by any number formed from the units $w_1, \cdots, w_{n-\delta-1}$; $\tau_1, \cdots, \tau_{\delta-1}$, is still $v^{n-\delta} = 0$ or $v^{(n-1)-(\delta-1)} = 0$. So our system in $n - 1$ units is of deficiency $\delta - 1$. Hence every number system of the type considered in n units, of deficiency δ, can be obtained by taking each independent number system in $n - 1$ units of deficiency $\delta - 1$ in the $w - \tau$ form and bordering it, as indicated, by a row and column τ_δ. There will be certain parameters introduced in this bordering which can be restricted by application of the associative law and by the fact that $\tau_\delta{}^{n-\delta} = 0$.

Professor Goursat's paper, which will be published in the *Transactions*, is in abstract as follows: The proof of Cauchy's integral theorem

$$\int_C f(z)dz = 0$$

* *Math. Annalen*, vol. 39.

which Professor Goursat obtained in 1883 (cf. *Acta Mathematica*, vol. 4 (1884), pp. 197–200; Harkness and Morley, Theory of Functions, p. 164) depends on the possibility of dividing up the region A, along whose contour C the integral $\int_C f(z)dz$ is to be extended, into small squares, or pieces of squares, in such a manner that the sum of the integrals $\int f(z)dz$ taken along each of these little contours becomes less in absolute value than an arbitrarily small positive quantity. The value of this sum is precisely $\int_C f(z)dz$, and hence this integral vanishes. The proof turns on the fact that, ε being a positive quantity chosen at pleasure, the squares can be taken of such a size that, z_i being a point properly chosen within or on the boundary of the ith square and z being any point on the boundary of this square, the relation

$$|f(z) - f(z_i) - (z - z_i)f'(z_i)| \leqq |z - z_i| \varepsilon$$

will be satisfied. In the present paper Professor Goursat shows that this relation can be established by the aid of the *sole assumption that $f(z)$ possesses at each point of the region in question a derivative, i. e.*, that

$$\frac{f(z_0 + h) - f(z_0)}{h}$$

converges toward a finite limit at the point z_0, which is any point of the region. From Cauchy's integral theorem follows at once Cauchy's integral formula

$$f(x) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{z - x},$$

and on these two theorems the whole theory of functions of a complex variable can be built up. Thus the question of whether in the definition of an analytic function it is necessary to require, in addition to the *existence* of a derivative, that the derivative shall be *continuous*—a question which up to the present time has remained open—is answered in the negative; the continuity of the derivative follows from its existence. The method employed in the proof is the familiar one of assuming that for a given square (or piece of a square) the relation is not satisfied, and then of subdividing the square. The assumption that a finite number of repetitions of the process of subdivision will not yield squares for each of which the relation can be satisfied leads to the conclusion that there exists a point z_0 within or on the boundary of the original square such that for some contours in every neighborhood of z_0 the relation cannot be satisfied, and here is a contradiction.

The theory of the function $\Gamma(1 + x)$ can be used to obtain the values of certain classes of definite integrals; these classes include some of those obtained in very artificial ways in the ordinary text-books. Professor Morley's note exhibits the process in the case of the integral

$$\int_0^{\pi/2} (\log 2 \cos \varphi)^m \varphi^n d\varphi$$

where m is any positive integer, n any positive even integer.

Professor Brown's paper presented an interesting illustration of a special form of tidal motion, supplementing a passage in the author's review of Darwin's work on Tides in the May number of the BULLETIN.

In his article in the *Mathematische Annalen*, vol. 39, Scheffers has given general methods by which the non-quaternion number systems of ranks 2 and 5 in six units can be determined. Dr. Strong has determined the remaining systems in six units.

Professor Lovett's paper studies the theory of curves of multiple curvature by the methods of the intrinsic geometry as developed recently by Cesàro. A line is said to be of triple curvature when no five consecutive points of it lie in the same linear space of three dimensions. The right line determined by two consecutive points, the plane determined by three consecutive points, and the linear space of three dimensions S_3 determined by four consecutive points are called respectively tangent, osculating plane, and osculating space. At every point of the curve there are ∞^2 normals; these all lie in the same S_3, which is called the normal space. Among these ∞^2 normals one is found in the osculating plane; it is called the principal normal. ∞^1 of the normals are perpendicular to the osculating plane; they are called binormals because they are perpendicular to two consecutive tangents. One only of the binormals is contained in the osculating space; it is called the principal binormal. The only binormal perpendicular to the osculating space is called the trinormal because perpendicular to three consecutive tangents. The tangent, trinormal, principal binormal, and principal normal at the same point are the principal directions of the curve at this point. The tetrarectangular tetrahedroid whose edges are the principal directions is taken as the fundamental tetrahedroid movable with the point. The following fundamental formulæ are derived and employed to investigate a curve at any one of its points :

$$\frac{dx}{ds}=\frac{t}{\rho}-1,\quad \frac{dy}{ds}=\frac{z}{R},\quad \frac{dz}{ds}=\frac{t}{r}-\frac{y}{R},\quad \frac{dt}{ds}=-\frac{x}{\rho}-\frac{z}{\gamma};$$

they express the necessary and sufficient conditions that a point in space remain fixed when the fundamental tetrahedroid is given an infinitesimal displacement along the curve. The analogous formulæ for the invariance of a direction whose cosines are α, β, γ, ε are

$$\frac{d\alpha}{ds}=\frac{t}{\rho},\quad \frac{d\beta}{ds}=\frac{\gamma}{R},\quad \frac{d\gamma}{ds}=\frac{\varepsilon}{r}-\frac{\gamma}{R},\quad \frac{dt}{ds}=-\frac{\alpha}{\rho}-\frac{\gamma}{r}.$$

In these formulæ ρ, γ, and R measure respectively the rapidity of the deviations of the curve from its tangent line, its osculating plane and its osculating space, respectively.

These relations are used to demonstrate that a triad of independent equations

$$f(s,\ \rho,\ \tau,\ R)=0,\quad g(s,\ \rho,\ \gamma,\ R)=0,\quad g(s,\ \rho,\ \gamma,\ R)=0$$

determines completely and uniquely a curve of triple curvature. The results of the integration yield in turn elegant expressions for the curvatures and principal directions at any point of the curve. The paper concludes with an extension of the barycentric analysis to a study of pentahedroidal potential. A novel theorem that may be mentioned in this abstract is that a curve of multiple curvature cuts its osculating space of highest dimensions or lies wholly on one side of that space according as the number of dimensions of the space necessary to the existence of the curve is odd or even. The paper will appear in the last number of the current volume of the *Annals of Mathematics*.

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