THE FEBRUARY MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

A REGULAR meeting of THE AMERICAN MATHEMATICAL SOCIETY was held in New York City on Saturday, February 25, 1899. Forty-seven persons were in attendance, including the following forty members of the Society:

Mr. Joseph Allen, Professor Maxime Bôcher, Professor E. W. Brown, Dr. A. S. Chessin, Dr. J. B. Chittenden, Professor F. N. Cole, Professor T. W. Edmondson, Professor H. B. Fine, Professor T. S. Fiske, Professor James Harkness, Dr. G. W. Hill, Mr. H. E. Hawkes, Professor Harold Jacoby, Mr. C. J. Keyser, Professor Pomeroy Ladue, Professor Gustave Legras, Dr. G. H. Ling, Professor E. O. Lovett, Dr. Emory McClintock, Professor James McMahon, Mr. James Maclay, Professor Alexander Macfarlane, Professor Mansfield Merriman, Dr. G. A. Miller, Professor Frank Morley, Professor W. F. Osgood, Professor A. W. Phillips, Professor James Pierpont, Professor M. I. Pupin, Professor J. K. Rees, Dr. W. M. Strong, Professor Henry Taber, Professor J. H. Tanner, Professor H. D. Thompson, Professor C. L. Thornburg, Professor E. B. Van Vleck, Professor J. M. Van Vleck, Professor A. G. Webster, Miss E. C. Williams, and Professor R. S. Woodward.

The President of the Society, Professor R. S. Woodward, occupied the chair. The meeting extended, as usual, through a morning and an afternoon session, which hardly sufficed, however, for the adequate presentation of the unusually large number of papers offered. The Council announced the election of the following persons to membership in the Society:—Mr. John B. Faught, Indiana University, Bloomington, Ind.; Mr. Edward B. Fishburne, Waynesboro, Va.; Professor William P. Graham, Syracuse University, Syracuse, N. Y.; Dr. Waldemar Schulz, Cornell University, Ithaca, N. Y.; Dr. Ernest J. Wilczynski, University of California, Berkeley, Cal. Four applications for membership were reported.

An amendment to the Constitution was declared adopted, by which retiring Presidents of the Society continue to serve in the Council for one year. It was announced that the summer meeting of the Society would be held at Columbus, Ohio, on Friday and Saturday, August 25–26, in affiliation with the American Association for the Advancement of Science.

The President then made, in behalf of the Council, an announcement of deep significance for the future of the Society. For nearly a year plans have been under discussion for providing adequate facilities for the publication of the constantly increasing number of valuable original mathematical papers produced in America. The committee appointed at the last summer meeting to consider this question has reported that it is desirable and feasible and in all respects for the best interests of mathematical science in this country that the Society should undertake the periodical publication of Transactions, beginning with January 1, 1900. This is a project which is ardently supported by many of the most prominent members of the Society. The general interest which it has aroused is already stimulating the Society to increased activity, the immediate effect being reflected in the attendance and number of papers presented at this meeting. The Council has appointed a committee, consisting of Professors T. S. Fiske, R. S. Woodward, E. H. Moore, M. Bôcher, and J. Pierpont, to secure the necessary financial guarantees for the proposed undertaking, the success of which is already well assured.

The following papers were presented at the meeting:

(1) Professor M. I. PUPIN: "Electrical oscillations on a loaded conductor."

(2) Professor MAXIME BÔCHER: "An elementary proof that Bessel's functions of the zeroth order have an infinite number of real roots."

(3) Professor J. M. PEIRCE: "Determinants of quaternions."

(4) Professor HENRY TABER: "The chief theorem of the theory of finite continuous groups."

(5) Professor ALEXANDER MACFARLANE: "On the imaginary of geometry."

(6) Professor W. F. OSGOOD: "On a means of generating a function of a real variable whose derivative exists for every value of the argument but is not integrable."

(7) Professor JAMES PIERPONT: "On arithmetizing mathematics."

(8) Professor E. O. LOVETT: "On a certain class of differential invariants."

(9) Dr. VIRGIL SNYDER: "Lines of curvature on annular surfaces having two spherical directrices."

(10) Professor E. W. BROWN: "On the progress of the calculations in the new lunar theory."

(11) Professor M. I. PUPIN: "Lagrange's equations and the principle of equality of action and reaction."

(12) Professor E. B. VAN VLECK: "On the determination of a series of Sturm's functions by the computation of a single determinant."

(13) Dr. G. A. MILLER: "On the primitive groups of degree 17."

(14) Dr. L. E. DICKSON: "Concerning the abelian and hypoabelian groups."

(15) Dr. F. H. SAFFORD: "Surfaces of revolution in the theory of Lamé's products."

(16) Dr. D. F. CAMPBELL: "On linear differential equations of the third and fourth orders in whose solutions exist certain homogeneous relations."

(17) Mr. E. R. HEDRICK: "On three dimensional determinants."

(18) Dr. G. H. LING: "An examination of groups whose orders lie between 1093 and 2000."

(19) Professor A. G. WEBSTER: "Traces illustrating the motion of the top."

In the absence of the authors the papers of Dr. Dickson, Dr. Safford and Dr. Campbell were read by title. Professor Peirce's paper was read by Professor Bôcher, and Dr. Snyder's by Dr. Miller. Mr. Hedrick was introduced by Professor Osgood. Dr. Campbell's paper was offered to the Society through Professor Bôcher. Abstracts of papers not intended for publication in the BULLETIN are given below.

Professor Macfarlane elucidates the theory of geometric imaginaries. In his paper on multiple algebra Professor Cayley identifies the imaginary of algebra with the imaginary of analytical geometry, and says: "In analytical geometry, without seeking for any real representation, we deal with imaginary points, lines, etc., depending on parameters of the form $a + bi$." The object of the present paper is to investigate the real representation of the so called imaginary points, lines, etc. Some authors, *e. g.*, Carr in his Synopsis, give a fictitious representation of the ideal intersection of a straight line and a circle, by drawing the supplementary hyperbola in a plane at right angles to the plane of the given circle; and it is supposed that the $\sqrt{-1}$ denotes the turning of the plane of the supplementary curve into the position at right angles to the original plane. There is no proof or application of the construction. It is not true, as Carr says, that the theory of quaternions removes all imaginariness from the symbol $\sqrt{-1}$. For Hamilton said: "It would be an error to confound geometrical imaginaries with those square roots of negatives for which the calculus of quaternions supplies from the outset a definite and real representation. M. Maximilien Marie in his representation does not go out of the original plane, but gets his *realized* curve by dropping the $\sqrt{-1}$ from the coördinates in which it appears. In the present investigation it is shown that when the roots of a quadratic equation are real, they are still capable of a true representation in the plane, that is, as complex quantities. They have then the same relation to hyperbolic trigonometry, that the pair of roots, when imaginary, have to circular trigonometry. The fundamental theorem for the complex angle is investigated; and it is shown that the pair of roots, when imaginary, may have a real linear representation, namely, as proportional to the cosine or the sine of a complex angle. The theory is then applied to the imaginary intersection of a straight line and circle, to drawing a tangent to a circle from an internal point, to the imaginary intersection of a straight line and equilateral hyperbola, to the intersection of two circles, to imaginary lines and imaginary circles.

The following is an abstract of Professor Osgood's paper: Let the function $f(x)$ be defined as follows:

$$f(x) = x^2 \sin \frac{1}{x}, \quad \text{when } x \neq 0; \qquad f(0) = 0.$$

Let $a_1, a_2, \cdots$ be the abscissas of maxima of the graph of $f(x)$, so taken that $a_1 > a_2 > \cdots > 0$. Form the function

$$\varphi_n(x) = f(x), \quad \text{when } 0 \leqq x \leqq a_n;$$

$$\varphi_n(x) = f(2a_n - x), \quad \text{when } a_n < x \leqq 2a_n.$$

The graph of this function in the interval $a_n \leqq x \leqq 2a_n$ is the reflection in the ordinate whose abscissa is a_n of the graph of $f(x)$ in the interval $0 \leqq x \leqq a_n$. Finally let

$$\omega_n(x) = \frac{1}{2a_n} \varphi_n(2a_n x), \quad \text{when } 0 \leqq x \leqq 1;$$

$$\omega_n(x) = 0, \quad \text{when } x > 1 \text{ or } x < 0.$$

This function, $\omega_n(x)$, whose graph is of the same *form* as the graph of $\varphi_n(x)$, but drawn to a different *scale*, if $2a_n \neq 1$, has for every value of x, a derivative, $\omega_n'(x)$, which is in general continuous, but in the neighborhood of the points $x = 0$ and $x = 1$ oscillates between limits nearly equal to $+1$ and -1.

Next consider the Cantor's set described in the *American Journal of Mathematics*, vol. 19 (1897), p. 167, §8; BULLETIN, November, 1898, p. 83. Let the set be placed in the interval $0 \leqq x \leqq 1$. Consider any one (the ith) of the 2^{n-1} subintervals (n); its length is denoted by l_n; let the abscissa of the extremity that lies nearer the point $x = 0$ be denoted by $x_n^{(i)}$. I now proceed to "fit the function $\omega_n(x)$ into this interval." I wish this expression to describe the process of forming the function

$$\Omega_n^{(i)}(x) = l_n \omega_n \left(\frac{x - x_n^{(i)}}{l_n} \right),$$

whose graph is of the same form as that of $\omega_n(x)$, but drawn to a different scale. This being done for each one of the intervals (1), (2), $\cdots$, a set of functions is obtained by means of which a function

$$F(x) = \sum_{n=1}^{\infty} \sum_{i=1}^{2^{n-1}} \Omega_n^{(i)}(x)$$

is defined, which has a finite derivative, $F'(x)$, for every value of x. The function $F'(x)$ is finite throughout the whole interval $0 \leqq x \leqq 1$, but near the extremities of the intervals (n) it oscillates between limits nearly equal to $+1$ and -1. These points form a set of positive content, and hence $F'(x)$ cannot be integrated *throughout the whole interval* (0, 1), though it can be integrated throughout certain parts of it.

It remains to "fit the function $F(x)$ into the free intervals" (1), (2), $\cdots$, its amplitude being simultaneously diminished. Let

$$F_n^{(i)}(x) = l_n^2 F \left(\frac{x - x_n^{(i)}}{l_n} \right),$$

$$F_1(x) = \sum_{n=1}^{\infty} \sum_{i=1}^{2^{n-1}} F_n^{(i)}(x);$$

and let $c_n = 10^{-n}$. Then the function $F(x) + c_1 F_1(x)$ is still integrable throughout certain intervals that lie within the former intervals (1), (2), $\cdots$. The function $F(x)$ must be "fitted into these intervals," its amplitude being diminished, and a function $F_2(x)$ formed in a manner similar to that in which $F_1(x)$ was just constructed. Proceeding in this way, we obtain a set of functions $F_k(x)$, $(k = 1, 2, \cdots)$, and the function

$$\Phi(x) = F(x) + c_1 F_1(x) + c_2 F_2(x) + \cdots$$

is the function required. It has a derivative for each value of x in the interval (0, 1). This derivative is finite throughout the interval; but it cannot be integrated throughout any portion of the interval.

Professor Lovett's paper constructs a class of $\frac{1}{2}n^2(n+1)$ second order differential invariants in n variables under the general projective group in n variables. These invariants are found in two ways: 1° by elimination, when the finite equations of the group are taken as starting point; 2° by integration, when the infinitesimal transformations alone are given, namely the integration of the complete system of partial differential equations formed by equating to zero the second extensions of the infinitesimal generators of the general projective group. For the values 1, 2, and 3 of n, the above system reduces to Schwarz's derivative, the invariants of Goursat, and those of Painlevé, respectively.

The results of Dr. Snyder's paper are summarized as follows: When the coördinates of the spheres which generate an annular surface satisfy two linear relations, all of the lines of curvature of the second system lie on spheres; these spheres can be determined by rational operations. The points of intersection of each such sphere with the circles of the first system lie in involution. The center of this involution is a plane curve, which touches a fixed circle in $2n$ points. All the lines of curvature pass through the "pinchpoints" and touch each other. Every such surface has one plane line of curvature of the second system; if it has two, all become plane and these planes all belong to the same axial pencil. Surfaces of revolution make one type of such surfaces. When the two directrices coincide, all of the spheres pass through a fixed circle; this is a line of curvature on the surface. In case the double directrix is a plane, a torsal line lies on the surface, which is a line of parabolic curvature. All the other planes containing lines of curvature pass through this line. The involution is now degraded. If the planes of a developable touch a fixed sphere, all its lines of curvature are rational; they lie on a family of spheres concentric with the fixed one. Cones form a particular example. The spherical line of curvature on the point locus of any complex is determined. The minimum line of curvature is determined for every surface.

There are five primitive groups of degree 17 that are included in the metacyclic group of this degree. In Jordan's enumeration of the primitive groups of degree 17 (*Comptes Rendus*, vol. 75, p. 1757) only one additional group besides the alternating and the symmetric is given. Dr. Miller finds that there are at least three such groups whose orders are 17.240, 17.480, 17.960. The first of these is a simple group and is included in each of the other two.

Dr. Dickson's paper gives the results of an investigation made a year ago which led to two new definitions of the abelian and the two hypoabelian linear groups. For the first definition, the author employs the conception of the group of isomorphisms of an *abstract* group, which was arrived at by Moore and Hölder independently. Consider the abstract group F generated by the operators θ, $A_1, \ldots, A_m, B_1, \ldots, B_m$ subject to the complete set of generational relations

$$\theta^p = A_i^p = B_i^p = 1, \quad \theta A_i = A_i\theta, \quad \theta B_i = B_i\theta \qquad (i=1, \cdots, m)$$

$$A_iA_j = A_jA_i, \quad B_iB_j = B_jB_i \qquad (i, j = 1, \cdots, m)$$

$$A_iB_i = \theta B_iA_i, \quad A_iB_j = B_jA_i \quad (i, j = 1, \cdots, m; i \neq j).$$

Limiting the discussion to the case $p =$ prime, every operator of F can be expressed in a single way in the form

$$\theta^t A_1^{x_1} B_1^{y_1} \cdots A_m^{x_m} B_m^{y_m},$$

the exponents t, x_i, y_i being taken modulo p. Introducing in the most general way a new set of generators θ', A_i', B_i' $(i = 1, \cdots, m)$ satisfying the above relations and capable of generating the entire group F, we obtain as the group of isomorphisms of F into itself a group meriedrically isomorphic with a $2m$-ary linear homogeneous group modulo p. For $p > 2$, the latter is the general abelian group; for $p = 2$, it is the first hypoabelian group.

Extending the group F by an operator J commutative with every operator of F and such that $J^p = \theta$, we obtain a group F_1 whose general operator can be expressed in a single way in the form

$$J^t A_1^{x_1} B_1^{y_1} \cdots A_m^{x_m} B_m^{y_m},$$

the exponents x_i, y_i being taken modulo p, while t is taken modulo p^2. The group of isomorphisms of F_1 is holoedrically isomorphic with the general abelian group on $2m$ indices taken modulo p. Similarly, a third abstract group F' is defined, whose group of isomorphisms is meriedrically isomorphic with the total second hypoabelian group if $p = 2$, and to a subgroup of the general abelian group if $p > 2$. Utilizing certain developments of Jordan on solvable groups (Traité, §§ 561 and 566), the author gives a concrete representation of the above groups F, F_1, and F' as elementary groups of linear homogeneous substitutions on p^m variables. The above results are then generalized so that we obtain, as groups of isomorphisms of certain very elementary groups, the general abelian and the first hypoabelian groups on $2m$ indices in the Galois field of order p^n.

The second definition of the general abelian linear group results from a simple application of the very fruitful concept of "the compound groups of a given linear homogeneous group" developed by the author in recent articles in the BULLETIN and in the *Proceedings of the London Mathematical Society*. THEOREM:—*The general abelian group on $2m$ indices is the largest $2m$-ary linear homogeneous group whose second compound possesses a certain linear invariant.*

The author then proves that the simple groups of order

$$\tfrac{1}{2}(p^{4n} - 1)\ (p^{2n} - 1)p^{4n}$$

derived from the decomposition of the abelian group on 4 indices in the $GF[p^n]$ and those derived from the decomposition of the orthogonal group on 5 indices in the $GF[p^n]$ are holoedrically isomorphic.

Wangerin* has treated the problem of obtaining the most general orthogonal surfaces of revolution such that, if Laplace's equation be written in coördinates corresponding to these surfaces, a solution may be obtained in the form of a Lamé's product with an extraneous factor. His result is that the meridian curves and the surfaces are of the fourth degree. Haentzschel† claims to have obtained surfaces of the thirty-second degree and denies the generality of Wangerin's result. In Dr. Safford's paper Haentzschel's surfaces are shown to be of the sixteenth degree only, and are resolved into four surfaces of the fourth degree. Thus Wangerin's result is proved to be correct and Haentzschel's criticism unfounded.

In the *Acta Mathematica*, volume 14 (1890), is a memoir by Brioschi in which he considers in a number of special

* *Berl. Monatsber.*, Feb., 1878.

† E. Haentschel, "Studien über die Reduction der Potentialgleichung, auf gewöhnliche Differentialgleichungen," Berlin, Reimer, 1893.

cases the question of homogeneous relations of degree m in the solutions of a linear differential equation of order n. He assumes that a relation $f_1(y_1, y_2, \cdots, y_n) = 0$ of degree m exists and then shows that a certain invariant must vanish. Instead of a relation, Dr. Campbell takes a form, $u(x) = f(y_1, y_2, \cdots, y_n)$ with arbitrary coefficients. Then instead of Brioschi's two equations of the form*

$$\varphi(\lambda) = 0, \quad \psi(\lambda) = 0,$$

we have, $$\varphi(\lambda) = \chi(u), \quad \psi(\lambda) = \varkappa(u),$$

where $\varphi(\lambda) = 0$, $\psi(\lambda) = 0$, $\chi(u) = 0$, $\varkappa(u) = 0$, would be linear differential equations in λ and u respectively. The following theorems can be established without much difficulty : 1° If ν and only ν linearly independent relations of the first degree exist among g functions, these g functions are the solutions of a linear differential equation of order $g - \nu$. 2° In the cases mentioned below, λ or $f(y_1', y_2', \cdots, y_n')$ cannot vanish identically. By means of these theorems Dr. Campbell finds, by an extension of Brioschi's method: (a) In case $m = 2$, $n = 3$, there cannot exist more than one relation. A necessary and sufficient condition that one relation exists is that the invariant a_3 vanishes. (b) In case $m = 3$, $n = 3$, there cannot exist more than three, or two and only two, linearly independent relations. A necessary and sufficient condition for the existence of three linearly independent relations is that the invariant a_3 vanishes ; and for one and only one relation, that the invariant Ω_{21} vanishes and a_3 does not. (c) In case $m = 2$, $n = 4$, there cannot exist more than three linearly independent relations. A necessary and sufficient condition for the existence of three linearly independent relations is that the invariants a_3 and a_4 vanish ; of two and only two linearly independent relations, that the invariants b_8 and Ω_{21} vanish and neither a_3 nor a_4 vanishes ; of one and only one relation, that the invariant Ω_{21} vanishes and b_8 does not vanish. In the course of the work appears the linear differential equation of which u is the general solution, in each of the above cases. Brioschi transforms the equation to a standard form when a relation exists. By means of this form, in each of the cases (a) and (c), the equation can be solved by solving equations of lower order. This has long since been known in case (a). In case (c) it can be shown without much difficulty that the equation can be solved by solving one linear differential equation of

* For the definitions of functions here used see the memoir above referred to.