CALCULUS OF FINITE DIFFERENCES.

Differenzenrechnung. Von A. A. MARKOFF, O. Professor an der kaiserlichen Universität zu St. Petersburg. Autorisierte deutsche Uebersetzung von THEOPHIL FRIESENDORFF und ERICH PRÜMM. Mit einem Vorworte von R. MEHMKE, O. Professor an der k. technischen Hochschule zu Stuttgart. Leipzig, B. G. Teubner, 1896. 8vo, iv + 194 pp.

THE number of books devoted exclusively to the calculus of finite differences has been comparatively few since its introduction as a branch of mathematics by Brook Taylor in his Methodus Incrementorum in 1715 and the publication of the first systematic treatise thereon by François Nicole in 1717. Perhaps one reason for this is the fact that the principles and theorems of the subject which are most frequently required in practical work are simple and can be proved when occasion for their use arises as, for instance, in interpolation and in the summation of series. The only works in English on finite differences now obtainable are Herschel's appendix to the translation of Lacroix's Differential and Integral Calculus published in 1816, which was followed by Herschel's Collection of Examples in 1820 ; the Calculus of Finite Differences incorporated in his volume on Differential Equations by John Hymers in 1839 which reached a second edition in 1858 ; the articles on finite differences in De Morgan's great work on the differential and integral calculus which appeared in 1842 ; and Boole's treatise which was published in 1860 and was intended as a sequel to his Differential Equations. A second edition of Boole's Calculus of Finite Differences, revised by J. F. Moulton, appeared in 1872. The latest works with which Professor Markoff's volume may be compared are Boole's treatise and Schlömilch's Theorie der Differenzen und Summen which was published in 1848.

Professor Markoff's work is characterized by a higher degree of generality in the theorems and greater thoroughness and rigor in the reasoning than is found in the writings of his predecessors. At the same time, his treatment is somewhat dry and severe, is rather too condensed at times, and is far from being as genial, interesting, and philosophical as that of Boole. The treatises of the two authors may be regarded as complementary. An important difference between their works is that Professor Markoff scarcely uses symbolic formulæ and eschews the symbolic method which was created by Lagrange and Laplace, and developed and employed with such signal success by Boole. Generating functions are not used in the work before us. The author nowhere refers to any analogies between the methods and results of the calculus of finite differences and the infinitesimal calculus. While some English mathematicians regard the latter as a branch of the former, many on the continent, perhaps more logically, consider that between the two calculi there is a great gulf fixed. With the exception of a few problems solved in the text, Professor Markoff's volume has the defect, usual in continental works, of not possessing exercises by which the student may test and strengthen his understanding of the subject. Throughout the book, h and not unity is taken as the difference between successive values of the variable.

The work consists of two equal parts, viz.: Part I, on Interpolation, and Part II, on Summation and difference equations. In Chapter I the general problem of interpolation is discussed, and Taylor's and Lagrange's formulæ are deduced. Limits to the errors made in using these formulæ are determined. The careful attention paid throughout the work to the limits of the errors in the formulæ employed is a feature in which this book is superior to others on the subject. In the search for an interpolation formula in which an additional term can be found without changing the preceding terms, Newton's formula for interpolation by equidistant intervals is obtained. In Chapter II finite differences are introduced for the purpose of calculating the coefficients in the Newtonian formula of interpolation. In Chapter III the expressions for differences in terms of differential coefficients, and for the latter in terms of the former, are derived. The treatment is rigorous but occasionally too brief for clearness. For example, the calculation of the table for the series of numbers $\Delta^m 0^n$ up to $m = n = 10$—this notation is not employed—requires more explanation than is given in the text. Chapter IV, on the construction and use of mathematical tables, supplies in part the lack of exercises in the preceding chapters. A commendable feature is the exact numerical calculation of the limits of error in connection with each example. Another important use of finite differences, namely, the application of interpolation to the approximate evaluation of definite integrals is shown in Chapter V. Among the interesting results obtained which may be new to some readers are the following: that a limit of the error involved in evaluating the integral $\int_c^d f(x)dx$ by the trape-

zoidal rule is
$$-\frac{(d-c)^3}{12s^2}f(\eta),$$

in which s is the number of parts into which the interval $d-c$ is divided, and η lies between c and d; and that the limit of error in evaluating the same integral by Simpson's parabolic rule is
$$-\left(\frac{d-c}{2}\right)^5 \cdot \frac{f^{\text{iv}}(\eta)}{90s^4},$$

in which s and η have the same signification as before. The Gaussian method of evaluating integrals is discussed in the latter part of the chapter. Chapter VI treats of Legendrean functions and develops the integral $\int_c^d \frac{dx}{z-x}$ as a continued fraction. The modern character of the work is well evidenced in Chapter VII, which is devoted to generalizations of some of the preceding theorems and formulæ.

Part II begins with Chapter VIII, on summation in its connection with the problem of the determination of a function from its difference of first order. Several series are summed. It is unfortunate that there is not something more on factorials, which here play an important part. Chapters IX and X discuss the Euler or Maclaurin sum formula,

$$\sum_a^b f(x) = \frac{1}{h}\int_a^b f(x)\,dx - \tfrac{1}{2}\{f(b)-f(a)\} + \frac{h}{12}\{f'(b)-f'(a)\} - \cdots,$$

and its applications. These chapters are exceedingly good. The treatment is better than can be found anywhere else and is all that can be demanded for the formula which Professor Sylvester considered the most important in the subject of finite differences. Bernoulli's numbers and Stirling's formula for $\log 1.2.3 \cdots (x-1).x$ are introduced in the course of the discussion. Chapters XI, XII, XIII are concerned with difference equations. On this topic there are marked differences in arrangement and treatment between Markoff and Boole. Difference equations of first order (Chapter XII) are not treated in as full and interesting a manner as in Boole; but the exposition on the general properties of difference equations (Chapter XI) and on linear difference equations with constant coefficients (Chapter XIII) is more satisfactory than that of the English author. Chapter XIV deals with the transformation of series. Applications are made to the evaluation of definite integrals and to the numerical calculation of slowly converging series by means of their transformation into series which are more rapidly convergent.

There are many references to the literature of the subject. No one interested in the calculus of finite differences can afford to be without Professor Markoff's valuable treatise.

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NOTES.

AT the meeting of the London Mathematical Society, held January 12th, the following papers were communicated: Linear transformation by inversion, by Dr. G. G. MORRICE; The zeros of the Bessel functions (second paper), by Mr. H. M. MACDONALD; A simple method of factorizing large composite numbers of any unknown form, by Mr. D. BIDDLE; On a determinant each of whose elements is the product of k factors, by Professor W. H. METZLER; Properties of hyperspace, in relation to systems of forces, the kinematics of rigid bodies, and Clifford's parallels, by Mr. A. N. WHITEHEAD; On the reduction of a linear substitution to its canonical form, by Professor W. S. BURNSIDE.

THE anniversary meeting of the Royal Astronomical Society was held February 10th. The retiring president, Professor R. S. BALL, delivered the address on presentation of the gold medal which had been awarded to Mr. F. McCLEAN. The president for the coming year is Professor G. H. DARWIN, and the honorary secretaries are Messrs. F. W. DYSON and F. NEWALL.

THE committee of organization of the international congress of mathematicians to be held in Paris from August 6th to 12th, 1900, has issued a preliminary circular requesting every one interested to notify the committee, whose headquarters are at 7 rue des Grands-Augustins, Paris, as to the probability of his attending or that of his not attending the congress. It is further requested that this intelligence be sent at once and that it specify the number of