passing through the sextic s' whose parameters are $\varkappa'$, λ', μ', ν'. Hence: *Any two curves of the above mentioned family of sextics lie on a cubic surface.*

As a special case we have for $\varkappa' = \varkappa$, $\lambda' = \lambda$, $\mu' = \mu$, $\nu' = \nu$ the theorem of Steiner: *A cubic surface can be inscribed to the Hessian F along any sextic curve of the family.*

CORNELL UNIVERSITY, ITHACA, N. Y.

ON THE SIMPLE ISOMORPHISMS OF A HAMILTONIAN GROUP TO ITSELF.

BY DR. G. A. MILLER.

(Read before the American Mathematical Society at its Fifth Summer Meeting, Boston, Mass., August 20, 1898.)

IF all the operators of any group G_1 of a finite order g_1 are transformed by each of the operators of one of its subgroups they are permuted according to a substitution group that has a 1, α isomorphism to this subgroup. The value of α will, in general, be different for the different subgroups and it will assume its maximum value α_1 when G_1 is transformed by all of its operators. The α_1 operators of G_1 that are commutative to each one of its operators constitute an abelian characteristic subgroup of G_1. Hence the factors of composition of G_1 are the prime factors of α, together with the factors of composition of the corresponding quotient group I_1.

I_1 is evidently simply isomorphic to the substitution group which is formed by all the permutations of the operators of G_1 when every operator of G_1 is transformed by each one of its operators. This substitution group must always be intransitive, since each operator is commutative to itself and hence the substitution group cannot contain any substitution that involves all the elements of the group. When G_1 is a simple group each of the transitive constituents of this substitution group must be simply isomorphic to G_1.

From the fact that the given substitution group which is simply isomorphic to I_1 must be intransitive and does not contain any substitution whose degree is equal to the degree of the group we may easily derive some important group properties of any non-abelian group. For instance, every non-abelian group must contain more than one system of conjugate operators.* If each system includes only two operators the group must contain at least three systems. There is no group in which each operator permutes the operators of no more than one system of conjugate operators. If a group contains only two systems of conjugate operators it must permute at least one of them according to a non-regular group.

I_1 is generally called the group of cogredient isomorphisms of G_1. If we transform all the operators of I_1 by each one of them we obtain its group of cogredient isomorphisms or the second group of cogredient isomorphisms I_2 of G_1. Similarly we may obtain the third group of cogredient isomorphisms I_3 of G_1, etc.

If we represent the orders of these successive groups of cogredient isomorphisms by $i_1, i_2, i_3, \cdots$, respectively, it must happen that for some finite value of a

$$i_\alpha = i_{\alpha+1}\,.$$

As i_α must then be equal to all the i's which follow it, we may suppose that it terminates the series. The groups for which $i_\alpha = 1$ are a special class of solvable groups which include the groups whose order is a power of a prime, according to a well known theorem of Sylow. Some of the properties of these groups have recently been studied by Ahrens† who also observed that they include the Hamiltonian groups as a very special case. It should be observed that the successive groups of cogredient isomorphisms correspond to Lie's "adjungierte Gruppen."

It may happen that there are operators that transform the operators of G_1 according to substitutions that are not found in the given substitution group which is simply isomorphic to I_1. All the substitutions that can be found in this way generate a group which is simply isomorphic to the group of isomorphisms of G_1. The group of cogredient isomorphisms is evidently a selfconjugate (not always characteristic)‡ subgroup of the group of isomorphisms whenever these two groups are not identical.

The simplest Hamiltonian group is the well known

---

* In this paper each *system* of conjugate operators is supposed to contain more than one operator unless the contrary is stated. That a group may contain only one system of conjugate subgroups is evident from the metacyclic groups of order $p\ (p-1)$ and degree p, p being any prime number.

† Ahrens, *Leipziger Berichte* (1897), vol. 49, pp. 616–626.

‡ Hölder, *Math. Annalen* (1893), vol. 43, p. 330.

quaternion group.* Its group of cogredient isomorphisms is the four group. Hence in this case $g_1 = 8$, $i_1 = 4$, $i_2 = 1$. The group of isomorphisms of the quaternion group may be represented as a transitive substitution group of degree 6 and order $6.4 = 24$. As this group must be isomorphic to the symmetric group of degree three, it must be simply isomorphic to the symmetric group of degree four. The group of isomorphisms of the quaternion group is therefore a complete group.

The direct product of any abelian group which contains no operator whose order is divisible by four and the quaternion group is clearly a Hamiltonian group and every Hamiltonian group may be represented as such a direct product.† We proceed to inquire into the group of cogredient isomorphisms of a direct product. If we represent by a given letter each of the operators besides identity of the different groups which are multiplied to form a direct product, each of the operators of the product is completely determined by the set of letters that belongs to its factors. To the group of cogredient isomorphisms of the product there evidently corresponds a simply isomorphic intransitive substitution group in these letters. As the order of this substitution group is equal to the product of the orders of the groups of cogredient isomorphisms of the given factors we have the following

Theorem I. *The group of cogredient isomorphisms of a direct product is the direct product of the groups of cogredient isomorphisms of the groups which were multiplied to obtain the first direct product.*

Corollary. The group of cogredient isomorphisms of any Hamiltonian group is the four group and its second group of cogredient isomorphisms is identity.

The group of isomorphisms of any Hamiltonian group presents much greater difficulties than that of cogredient isomorphisms. We shall first consider a general principle. When a group contains a characteristic subgroup‡ and admits isomorphisms to itself which do not affect the operators of the quotient group with respect to this characteristic subgroup, the operators of the group of isomorphisms which correspond to these isomorphisms must form a selfconjugate subgroup. This follows directly from the fact that all the simple isomorphisms of the group to itself must correspond to simple isomorphisms of the given quotient group to itself.

* Dedekind, *Math. Annalen* (1897), vol. 48, p. 548.
† Miller, BULLETIN (1898), vol. 4, p. 510.
‡ Frobenius, *Berliner Sitzungsberichte*, 1895, p. 183.

This principle may be stated in a somewhat more general form as follows: When a group contains a selfconjugate subgroup, and admits isomorphisms to itself which do not affect the quotient group with respect to this selfconjugate subgroup, the corresponding operators of the group of isomorphisms must form a subgroup that is transformed into itself by all the operators of the group of isomorphisms which correspond to isomorphisms that make the given selfconjugate subgroup simply isomorphic to itself. When this selfconjugate subgroup is characteristic we obtain the principle as stated above.

The commutator subgroup of any Hamiltonian group H is of order 2. With respect to this characteristic* subgroup H is isomorphic to an abelian group. According to the principle which has just been stated, the operators of the group of isomorphisms of H, which do not affect the operators of this abelian quotient group, form a selfconjugate subgroup. This selfconjugate subgroup must always include the group of cogredient isomorphisms, and each of its operators, besides identity, must be of order two. In the quaternion group it is identical to the group of cogredient isomorphisms.

We shall now inquire into the order of this selfconjugate subgroup when H is any Hamiltonian group of order $2^\alpha p_1^{\alpha_1} p_2^{\alpha_2} \cdots$; $p_1, p_2, \cdots$ being any odd prime numbers. If one operator in a division with respect to the commutator subgroup is of an odd order the other must be of an even order. Hence we may confine our attention to the given isomorphisms of the subgroup of order 2^α to itself. Since the number of these isomorphisms is $2^{\alpha-1}$ we have

Theorem II. *If the order of a Hamiltonian group is of the form* $2^\alpha p_1^{\alpha_1} p_2^{\alpha_2} \cdots$, $p_1, p_2 \cdots$ *being odd prime numbers, the order of its subgroup of isomorphisms which do not affect the operators of its quotient group with respect to the commutator subgroup is* $2^{\alpha-1}$. *This subgroup of isomorphisms contains* $2^{\alpha-1} - 1$ *operators of order two and it is selfconjugate in the group of isomorphisms of the given Hamiltonian group.*

Corollary. When the order of a Hamiltonian group is not divisible by 16 *this selfconjugate subgroup of isomorphisms is identical to the group of cogredient isomorphisms.*

Any Hamiltonian group of the given order contains a characteristic subgroup of order $2^{\alpha-2}$. The simple isomorphisms of the group to itself which do not affect the operators of the quotient group with respect to this characteristic subgroup must correspond to a selfconjugate subgroup

* Miller, BULLETIN (1898), vol. 4, p. 135.

in the group of isomorphisms of the Hamiltonian group. When $a = 3$ this reduces to the case which has just been considered.

We proceed to consider the group of isomorphisms of a group G which is the direct product of a series of subgroups G_1, G_2, G_3, $\cdots$, G_a such that each of these subgroups corresponds to itself in every simple isomorphism of G to itself. We may suppose that each operator of these subgroups with the exception of unity is represented by a particular letter. Each operator of G will then be represented by a certain combination of these letters. Since each of the given subgroups can be made simply isomorphic to itself in every possible manner without affecting the isomorphism of the other subgroups, the group of isomorphisms of G may be represented as the product of the substitution groups corresponding to the simple isomorphisms of each of these subgroups to itself. Hence

Theorem III. *If a group is a direct product of a series of subgroups such that each of these subgroups corresponds to itself in every possible simple isomorphism of the group to itself then the group of isomorphisms of this group is the direct product of the groups of isomorphisms of the given subgroups.*

Corollary. The group of isomorphisms of a Hamiltonian group is the direct product of the groups of isomorphisms of its subgroups of orders 2^a, $p_1^{a_1}$; $p_2^{a_2}$, $\cdots$.

CHICAGO, *August*, 1898.

GALOIS'S COLLECTED WORKS.

Oeuvres mathématiques d'Évariste Galois; publiées sous les auspices de la Société Mathématique de France, avec une introduction par M. ÉMILE PICARD. Paris, Gauthier-Villars et Fils, 1897. 8vo, x + 63 pp.

MATHEMATICIANS will welcome this new edition of Galois's works published under the auspices of the French Mathematical Society. They were originally collected and published in 1846 by Liouville in the *Journal de Mathématiques.* The present edition is a reprint of the first. It is accompanied by a portrait* of Galois and an introduction by Picard.

---

* A note stating that this portrait was made *d'après nature* when Galois was fifteen or sixteen years of age should have been added. It was in the possession of the family when discovered by Dupuy.